On the Need of Convexity in Patchworking

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Abstract

Viro's construction of real smooth hypersurfaces uses regular (also called convex or coherent) subdivisions of Newton polytopes. Nevertheless, Viro's construction (sometimes called *patchworking*) can be applied as well to arbitrary subdivisions as a purely combinatorial procedure. Are the schemes coming from non-regular subdivisions, still topological types of some real smooth hypersurfaces? In the first part of this paper we prove a combinatorial version of Hilbert's Lemma (a consequence of Bezout's Theorem) that bounds the depth of nests in a T-curve, and we use this result and previous work by I. Itenberg to answer the question affirmatively for T-curves of degree less than six.

According to V.A. Rokhlin [18], a real algebraic scheme has complex orientation of type I (alternatively of type II) if any curve with this real scheme divides (does not divide) its complexification. A real algebraic scheme has *indefinite type* if there are type I and type II curves with that particular scheme. In the second part of this paper, we describe a combinatorial algorithm, due to I. Itenberg and O. Viro, that allows one to determine the type of a T-curve. We then present a partial list of indefinite schemes for T-curves of degrees 7 and 8.

1 Introduction

In this paper we are interested on the construction of real algebraic smooth plane projective curves (smooth real curves, for short) of fixed degree d with prescribed topology. In particular we analyze a combinatorial construction due to Oleg Viro [21]. Smooth real algebraic curves are defined by smooth real homogeneous polynomials in three variables of degree d considered up to multiplication by a non-zero real number. Such polynomials have a *real zero locus* which is the disjoint union of circles embedded in $\mathbb{RP}^2$. We call the *real scheme* $\mathbb{R}C$ of an algebraic curve the embedding of the components of the real zero locus inside $\mathbb{RP}^2$. Hence the real scheme refers to the particular disposition of the components of the real zero locus.

Hilbert's 16th problem concerns the classification of real schemes of smooth curves of degree d. As in any classification, it is important to have efficient methods for the constructions of realizable elements. The classical methods of construction of real smooth curves introduced by Harnack and Hilbert remained the only construction strategies until Viro proposed a new combinatorial method for constructing curves (see [17, 21, 23], or [6, Chapter 11]). Viro's method uses regular subdivisions of Newton polytopes and the theory of toric varieties to construct a *T-curve*: a piecewise linear curve in a discrete model of $\mathbb{RP}^2$. Viro's method has been used to successfully complete the classification of smooth real curves of degree 7 [21], the construction of counterexamples to the longstanding Ragsdale conjecture [7, 8], the construction of smooth real surfaces [3, 9, 12] and the construction of complete intersections with prescribed to topology [11, 20]

We now recall the construction of T-curves (we recommend [13] for a nice introduction). Let T be a triangulation of the triangle $Q_d := \{x \geq 0, y \geq 0, x + y \leq d\}$ whose vertices are the integral lattice points lying inside Q_d. Assign an arbitrary choice of signs to each of the lattice points. Create three copies $s_x(T), s_y(T)$ and $s_x(s_y(T))$ of T via the reflections with respect to the coordinate axes. Define $\mathcal{T}$ to be $T \cup s_x(T) \cup s_y(T) \cup s_{xy}(T)$. This is the four-fold symmetric triangulation of the $d \times d$ diamond. Glue the sides of the regions and also antipodal points of the

boundary of the diamond $\mathcal{T}$. The resulting polyhedral surface is homeomorphic to $\mathbb{RP}^2$; we denote it by $P(T)$. We will call the boundary of the diamond the *line at infinity*.

Extend the original distribution of signs to the other three quadrants as follows: If $\operatorname{sgn}(i,j)$ is the sign of point (i,j) inside Q_d, then the sign of $(-i,j)$ in $s_x(T)$ is $(-1)^i \operatorname{sgn}(i,j)$, the sign of $(i,-j)$ in $s_y(T)$ equals $(-1)^y \operatorname{sgn}(i,j)$ and the point $(-i,-j)$ will have sign $(-1)^{i+j} \operatorname{sgn}(i,j)$. From this triangulation and distribution of signs, we may construct a piecewise linear curve. If a triangle of the triangulation has different signs on its vertices, divide the vertices with a midline. Consider the piecewise linear curve obtained as the union of these segments and contained in the region $\mathcal{T} = T \cup s_x(T) \cup s_y(T) \cup s_x(s_y(T))$. Since antipodal points are identified, the curve is the disjoint union of circles. This is a T-curve.

This process of construction has been called *patchworking* [13, 23]. We say that a triangulation is *convex* or *regular* if there exists a convex piecewise-linear function from T to $\mathbb{R}$, which is linear on each triangle of T and not linear on the union of any two triangles of T. For more information on triangulations, especially regularity of triangulations see [2, 6, 14, 26]. The main result of Viro about T-curves is the following:

Theorem 1.1 (Viro's patchworking theorem) *Under the assumption that the triangulation T is convex, for each T-curve in $P(T)$, there exists a non-singular real algebraic plane projective curve and a homeomorphism of the $\mathbb{RP}^2$ onto $P(T)$ mapping the set of real points of this curve onto the T-curve.*

The proof of Viro's theorem uses the convexity of the triangulation to guarantee that the T-curve is isotopic to the real zero locus of a certain family of real smooth curves, but the construction can be applied as an abstract combinatorial operation to arbitrary triangulations.

QUESTION: *Do combinatorial T-curves constructed from non-regular triangulations, still yield schemes of real smooth algebraic curves?*

Several reasons support the investigation of the above open question: First, it is sometimes a laborious process to corroborate that a triangulation is convex, since one must solve a large system of inequalities. Second, there is evidence that regular subdivisions may be scarce inside the set of all triangulations [1, 4]. Third, Recently F. Santos[19] has found a new family of counterexamples to Ragsdale conjecture based on a family of **non-regular** triangulations. It is very interesting to note that Santos's counterexamples improved the counterexamples given by Itenberg and Hass. The asymptotic value of the number of positive ovals divided by the square of the degree of the curves is $13/16 = 0.8125$ in Itenberg's counterexamples, for Hass's counterexamples the asymptotic value is $5/6 = 0.83333$, while Santos's counterexamples attains $17/20 = 0.85$. Note that Petrowsky's inequality establishes the $7/8 = 0.875$ is a limit for the increment of the ratio.

In addition, it is well-known that the real algebraic variety defined by a smooth polynomial of degree d on $\mathbb{RP}^2$ has at most $(d-1)(d-2)/2+1$ components and that this bound is tight (for example, see [22]). Itenberg [9] provided a combinatorial proof of Harnack's inequality. Although he does not mentioned this, his proof remains valid for Viro's construction using even non-convex triangulations. S. Orevkov (private communication) has also proved that the Petrowsky's inequalities are true for T-curves arising from arbitrary triangulations. We have accumulated considerable experimental evidence, based on the computer program `Pisces` [5, 24], that indicates Arnol'd inequalities (see [25]) hold true as well. In fact we have observed several well-known restrictions on the Betti numbers of smooth surfaces, such as Smith-Thom and Commesatti-Kharlamov inequalities hold as well for arbitrary triangulations of higher dimension.

If there are schemes of T-curves coming from non-regular triangulations which are not schemes of real algebraic curves then they must violate some of the topological restrictions for real algebraic curves (see [22] for a survey of currently known prohibitions). In particular, as corollaries of Bezout's theorem, there are some natural restrictions for the nestings of real algebraic schemes of degree d. For instance,

for a curve of degree $2k$, the number of ovals in a nest, or in two disjoint nests, is at most k. This has sometimes been called Hilbert's lemma [25, p. 54]. In Section 2 we see that the following theorem is true for arbitrary triangulations.

Theorem 1.2 (Combinatorial Hilbert's Lemma) *Given any sign vector on any triangulation of a degree d model of $\mathbb{RP}^2$, the depth of a nest in the associated T-curve is no more than $d/2$. Moreover, if there is a pair of nests in the T-curve, the sum of their depths is less than or equal to $d/2$.*

This is listed by Viro as a prohibition of the topology of real algebraic curves that is not necessarily valid for flexible curves [22, §2.5]. From this result and the combinatorial proof of Harnack's inequality due to I. Itenberg [9] we obtain the following corollary:

Corollary 1.3 *All combinatorial schemes of T-curves arising from arbitrary triangulations of degree less than or equal to 5 are schemes of real algebraic smooth curves.*

Now we establish some notation and terminology. A planar triangulation is *primitive* when all its triangles have area $1/2$. For planar triangulations this is equivalent to the triangulations using all of the integer lattice points of Q_d as vertices. We observe that each component of a smooth real curve, as well as for T-curves, is homeomorphic to a circle, and that two embeddings are possible for a circle in $\mathbb{RP}^2$. If the complement of the circle is connected, we say it is a *pseudoline*. The complement of the circle may be disconnected, consisting of a circle and a Möbius band, and this case the component is called an *oval*. The *interior of an oval* $\mathcal{O}$ is the connected component of $\mathbb{RP}^2 \setminus \mathcal{O}$ that is homeomorphic to a disc. The other component is called the *exterior of the oval*. A pair of ovals is *nested* when one of the ovals is in the interior of the other. A set of h ovals such that every pair of ovals is nested will be called a *nest of depth h*.

2 The Convexity Hypothesis for Patchworking

Bezout's theorem is essential to restrict the possible topologies of smooth curves (see [22, §1.3]). For instance, relying on the fact that a line intersects with a degree d curve in at most d points, we can guarantee that the depth of a nesting of ovals is no more than $d/2$. In fact, more sophisticated restrictions on the topology could be derived this way, from the intersection with higher degree curves. The question arises whether the abstract T- curves, constructed over non-regular triangulations satisfy, in some sense, Bezout's theorem or its consequences. In this section we settle Theorem 1.2. We will require a few preliminaries on the theory. We remind the reader that within the family of *regular* T-curves, all of these results are known to be valid as a corollary of Viro's patchworking theorem. For reference the three sides of the triangle Q_d will be called the x-axis, the y-axis, and the $(x+y)$-axis.

Definition 2.1 *Let Γ be a path of edges, with consecutive vertices $(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$. We say that Γ is* pseudo-horizontal *if the sequence $i_1, i_2, \ldots, i_k$ is strictly monotone. We say that Γ is* pseudo-vertical *if the sequence $j_1, j_2, \ldots, j_k$ is strictly monotone. Similarly, we say that Γ is* pseudo-diagonal *if the sequence $i_1 + j_1, i_2 + j_2, \ldots, i_k + j_k$ is strictly monotone. See Figure 1 for an example.*

Definition 2.2 *Let T be a triangulation of the triangle $Q_d := \{x \geq 0, y \geq 0, x+y \leq d\}$. We say that a subgraph of the triangulation T is a* Y-subgraph *(see Figure 1 for the reason of this name) if it is the union of three paths $\Gamma_x, \Gamma_y, \Gamma_{x+y}$ that satisfy:*

1. *The paths start at a common vertex v, called the* junction point of the Y-subgraph.
2. *The end point of Γ_x is the x-axis, the end point of Γ_y is the y-axis and the end point of Γ_{x+y} is the line $x + y = d$.*
3. *The path Γ_x is pseudo-horizontal, the path Γ_y is pseudo-vertical and the path Γ_{x+y} is pseudo-diagonal.*

We remark that we allow in our definition the possibility that one of the paths is of length zero, which means the junction point v is on one of the boundaries of the triangle Q_d. Also, notice that as a consequence of the last property, if the junction point of the Y-subgraph is at the vertex (i, j), then the number of edges along the path Γ_x is smaller or equal than i, the number of edges along the path Γ_y is no more than j and the number of edges along the path Γ_{x+y} is no more than $d - (i + j)$. Hence the total length of the Y-subgraph is no more than d.

Proof of Theorem 1.2: Let T be a triangulation of the triangle $Q_d := \{x \geq 0, y \geq 0, x + y \leq d\}$. Recall that d is the degree of the curve and that the vertices of the triangulation are lattice points within Q_d. T is duplicated three times to obtain $\mathcal{T}$, a triangulation of the projective plane $P(T)$. We will show that given any pair of vertices of $\mathcal{T}$, say w_1 and w_2, there exists a piecewise-linear pseudoline inside $P(T)$, composed of at most $2d$ edges, that joins the two vertices and intersects any T-curve transversally in at most d points. Obviously, as in the algebraic Bezout's theorem, this will imply the desired theorem: if w_1, w_2 are points inside the innermost ovals of two distinct nests, then the pseudoline crosses each oval of the nests twice. Thus no more than $\lfloor d/2 \rfloor$ ovals are possible.

If w_1 and w_2 both belong to one of the axis lines or to the line at infinity, then take that line as the pseudoline. For the general case, we wish to construct a pseudoline by using the four-fold symmetry of $P(T)$. Let v_1, v_2 be the images of the vertices w_1, w_2 that lie inside the original triangle Q_d (all vertices have four images). By Lemma 2.4, there exists a Y-subgraph G, that contains both v_1 and v_2. From the Y-subgraph we may construct a pseudoline containing w_1 and w_2 and from the length properties of the Y-subgraph this pseudoline will have length at most $2d$. The construction is as follows: Consider all symmetric copies of the above Y-subgraph under the reflections s_x and s_y (see Figure 2). There are four possible pseudolines that can be built from these 12 paths, namely, (a, b, b', b'', a''), (a, b, b''', b'', a''), (a', b', b, b''', a'''), and $(a', b', b'', b''', a''')$. Each one spans three consecutive quadrants of $P(T)$. Select one that touches the two points in w_1 and w_2.

To complete the proof of Theorem 1.2, let w_1 and w_2 be two vertices of $\mathcal{T}$. We just saw that there is a pseudoline with length at most $2d$ and, by construction, the pseudoline is composed of egdes of $\mathcal{T}$, and each edge has a symmetric copy that is also in the pseudoline. If the endpoints of an edge have coordinates of different parity, the edge or its copy are possibly touched by the T-curve, *but not both.* This is clear from the sign changing rule. If all edges of the pseudoline have this property, which is the case for primitive triangulations, we can guarantee the number of intersections is no more than half the edges along the pseudoline, which is no more than d.

The remaining issue is what happens to the "bad" edges for which both copies intersect the T-curve. They must be necessarily longer, spanning two or more horizontal, vertical, or diagonal levels. Let L_1 and L_2 be the number of good and bad edges respectively. We have that the length of the pseudoline is $2(L_1 + L_2)$ which is less than or equal to $2d - 2L_2$. The number of intersections of the pseudoline with the T-curve has to be no more than $L_1 + 2L_2$. From these we have $L_1 + 2L_2 \leq d - 2L_2 + 2L_2 = d$ □

Remark 2.3 We remark that purely a combinatorial notion of intersection of two T-curves has been stated within the context of regular mixed subdivisions by Sturmfels [20]. One can completely rewrite the proof of Theorem 1.2 using the language of mixed subdivisions. It is well-known that for algebraic curves, Bezout's theorem offers further restrictions from the intersection with higher degree curves; it would be desirable to know whether all of these restrictions can also receive combinatorial proofs. Note that Lemma 2.4 is a combinatorial analog of the algebraic statement *"given any pair of points, there is a line that contains them."* Such statements are not obvious in the combinatorial framework of arbitrary triangulations.

Lemma 2.4 *Let T be a triangulation of the triangle $Q_d := \{x \geq 0, y \geq 0, x+y \leq d\}$. Let v_1 and v_2 be two vertices of T. There exists a Y-subgraph containing both v_1 and v_2 as vertices.*

Proof: First, we will prove that the set of Y-shaped subgraphs of T that contain an arbitrary point v is not empty. If v is on the boundary of Q_d, then one of the axes contains it and can be taken as the Y-subgraph containing v. When v is a point in the interior of Q_d, there is always an edge in the star of v that joins v to a vertex with smaller x-coordinate, and another edge joining v with a vertex with smaller y-coordinate. Similarly, organize the lattice points of the triangle Q_d into parallel diagonal lines of minus one slope. Any given vertex of T lies in one of these lines, say $x + y = k$ for some $0 \leq k \leq d$. As we observed before for horizontal or vertical paths, there is always an edge leaving v that joins it to a vertex in a line $x + y = r$ with $r > k$. Based on these facts we can find a sequence of edges of the triangulation that form a "pseudo-horizontal" path, that passes through v and connects it to the y-axis, a "pseudo-vertical" that passes through v and ends at the x-axis and a "pseudo-diagonal" path that passes through v and ends at the $(x + y)$-axis.

A Y-subgraph breaks the triangle Q_d into three regions: The region YD containing the vertex $(0, d)$, the region XY containing the origin and the region XD containing the vertex $(d, 0)$. Note some of this regions may be empty when the join vertex of the Y-subgraph is along one of the axis (i.e. the Y-subgraph is a line).

Now, let us consider the family of Y-subgraphs $\Phi(v_1)$ containing the point v_1. We will prove that some member of the family contains the other point v_2. Suppose no member of $\Phi(v_1)$ contains v_2. For each member of $\Phi(v_1)$ the vertex v_2 lies within one of these regions. Let us choose Γ, a Y-subgraph in $\Phi(v_1)$, that leaves v_2 in a region of smallest number of triangles. We will show that we can change a few edges of Γ and obtain another member of $\Phi(v_1)$ such that there are fewer triangles in the region containing v_2, thus reaching a contradiction.

Since the remainder of the proof is long, we give the reader an indication of where we are going. First we will define what it means to have triangles "tilted away" from an axis. We then introduce discrete "homotopies" that allow us to replace a Y-subgraph in $\Phi(v_1)$ with a second Y-subgraph that encloses less area. We then

prove that if a Y-subgraph encloses v_2 in a region of minimal area, the triangles interior to the region that contain edges of the Y-subgraph must "tilt away" from the axes. This allows us to concentrate on the vertex where these tilted sequences of triangles meet. However, we then show that we can create a new Y-subgraph in $\Phi(v_1)$ that has less area than the supposedly minimal area. This contradiction will prove the theorem.

We will present the construction for the case when the region containing v_2 is XY. The other two cases can be written almost verbatim replacing "pseudo-vertical" by "pseudo-diagonal" or "tilted away from the x-axis" by "tilted away from the $(x+y)$-axis". For the skeptical reader we note that the case of having v_2 inside region XY is identical to the other two cases when we transform the triangle Q_d into the "homogeneous" triangle

$$HQ_d = \{(x,y,z) \mid x+y+z = d\} \cap \{(x,y,z), x \geq 0, y \geq 0, z \geq 0\} \subset \mathbb{Z}^3.$$

So the vertex (i,j) in Q_d is mapped to the vertex $(i,j,d-(i+j))$. The triangle HQ_d is now an equilateral triangle, the triangulation T of Q_d is combinatorially the same (adjacencies are the same) and the notion of being "tilted away" with respect to a given axis (see below) is invariant under this transformation.

Tilted sequences of triangles: We need the definition of a *tilted sequence of triangles*: Let Γ_x be a pseudo-vertical path starting at the x-axis. Let S be a sequence of triangles, all in the same side with respect to Γ_x and using consecutive edges of the path. We say that S is *tilted away from the x-axis* if the y coordinate of the third vertex (the one not on the path Γ_x) has y coordinate greater than or equal to the largest x-coordinate of the other two vertices (which, by assumption, are consecutive vertices on Γ_x). In a similar way for a path Γ_y starting at the y-axis, if S is a sequence of triangles, all in the same side with respect to Γ_y and using consecutive edges of the path, we say that S is *tilted away from the y-axis* if the x-coordinate of the third vertex has x coordinate greater than or equal to the largest x-coordinate of the other two vertices. See Figure 3 for an example. Finally, we say

a sequence of triangles along a path Γ_{x+y}, starting at the $x+y$ axis, is *tilted away from the $x+y$ axis* if the sum of the coordinates of the third vertex of each triangle, is smaller than or equal to the smaller of the sums of coordinates for the other two vertices.

Discrete homotopies that reduce area: Note that the paths Γ_x and Γ_y border the region XY containing v_2. We focus our attention on the triangles contained in XY that use the edges of $\Gamma_x \cup \Gamma_y$. First, there are some operations that in an obvious way could produce a new Y-subgraph that leaves v_2 in an XY region with smaller number of triangles. We represent them in the Figure 4. We call these reduction operations *homotopies*.

In part (A) of Figure 4, we present the situation in which the first triangle along the Γ_x or Γ_y paths has two of its vertices on the corresponding axis we can exchange the first edge on the path by another which cuts out a triangle from the region. Part (B) presents two consecutive triangles sharing an edge We can slide down the path cutting out those triangles because new two edges are also pseudo-horizontal (alternatively pseudo-vertical). Part (C) shows that if an edge that jumps two or more units is the support of a triangle whose third vertex has no bigger x-coordinate (correspondingly y-coordinate), then we can replace that edge with the two other edges of the triangle, preserving the property that the path is pseudo-horizontal (or pseudo-vertical). Similarly, Part (D) shows a triangle, two of its edges part of a path, we can replace them with the longer third edge. We are again reducing the number of triangles by one. We note that the presence of vertex v_1 or the junction point v may potentially halt the application of this homotopy, because the new Y-subgraph might no longer be a member of $\Phi(v_1)$ anymore as vertex v_1 is lost. We will see later that the homotopy operations can be modified for these cases.

Minimal Y-subgraph implies "tilted away": We will see now how we can still construct a new Y-subgraph in $\Phi(v_1)$ with smaller area in the region containing v_2. We break the proof into two cases.

Case I: The point v_1 is not a vertex in $\Gamma_x \cup \Gamma_y$. Hence v_1 lies on the path Γ_{x+y}. The triangles supported on the first edges of Γ_x and Γ_y must be tilted away from the corresponding axis, else we could apply homotopies of type (A). Edges that follow must also be tilted away otherwise a homotopy operation of type (B) or (C) could be applied. In this way, we continue to create two sequences of triangles that are tilted away from the x and the y axes and run along the paths Γ_x and Γ_y, respectively. Let (a, b, c) be the last triangle in the sequence of triangles tilted away from the y-axis. There are two possibilities to consider. Either the edges ab and bc are consecutive edges in $\Gamma_x \cup \Gamma_y$ (see Figure 5, part (i)) or only the edge ab is in path $\Gamma_x \cup \Gamma_y$ (see Figure 5, part (ii)).

Consider first situation (i). We can apply the homotopies, unless the junction point v equals b, since then a homotopy operation loses the point v_1 which is along the pseudo-diagonal path Γ_{x+y}. Hence, the edge bc is necessarily part of the pseudo-vertical path Γ_x. Note that at least one of the vertices, a or c, has the property that the sum of its coordinates is smaller than the sum of the coordinates of $b = v$. Otherwise, the triangle (a, b, c) is not contained in the region XY. In situation (i) when $b = v$, we simply replace the edges ab and bc with the edges ac and gb where g is a or c, and has smaller sum of its coordinates than $b = v$. This creates a new junction point g and a longer path Γ_{x+y} with one less triangle or area.

Now consider situation (ii). Note that the location of vertex c shown in part (ii) of Figure 5 is accurate: the triangle subsequent to (a, b, c), say (b, c, d), must be inside region XY. This is because a triangle (b, e, d), with $e \neq c$ either contradicts the fact that (a, b, c) is the last triangle tilted away from the y-axis, or we could apply a homotopy of type (C). Since the triangle (a, b, c) is the last triangle tilted away from the y-axis, the edge ab is pseudo-horizontal (it may also be pseudo-vertical, as the two properties are not exclusive).

First, suppose that ab and bd are contained in Γ_y. Then we can apply a homotopy of type (B), creating a new element of $\Phi(v_1)$ with smaller area. Second, suppose ab, bd are part of Γ_x or ab is in Γ_x and bd is in Γ_x (this last case is equivalent to

$b = v$). In both cases, the sequence of triangles tilted away from the x-axis has also a last triangle, which must come after (a, b, c) if you list triangles in order along the path $\Gamma_y \cup \Gamma_x$ going from the y-axis toward the x-axis. Let (a', b', c') be the last triangle in the sequence of triangles tilted away from the x-axis. Clearly the junction v is not a vertex of (a', b', c'). This is important, because we will be able to apply a homotopy around (a', b', c'): If both edges $a'b'$ and $b'c'$ belong to Γ_x we can apply homotopy operation of type (D). If only $a'b'$ is an edge of Γ_x then the vertex c' has larger y-coordinate than $a'b'$ (since the triangle is tilted away from the x-axis).

Thus the triangle (d', a', e') in region XY that precedes (a', b', c') when going from v toward the x-axis, is in fact the triangle (d', a', c'). Otherwise, we contradict the fact that (a', b', c') is the last triangle tilted away from the x-axis, or we could apply a homotopy of type (C). In conclusion we can apply the homotopy of type (B) to the triangles $(a', b', c'), (d', a', c)'$ and produce a new element of $\Phi(v_1)$ with fewer triangles in the XY region.

Case II: The point v_1 is along $\Gamma_x \cup \Gamma_y$. We can assume without loss of generality that v_1 belongs to Γ_x.

Again we can construct a sequence of triangles, tilted away from the y-axis. The sequence ends either at a triangle that has two edges along $\Gamma_x \cup \Gamma_y$ (see Figure 5 part (i)) or at two consecutive triangles (a, b, c) and (b, c, d) sharing a common edge, where (a, b, c) is the last triangle tilted away from the y-axis (see Figure 5, part (ii)).

Consider situation (i). If b is different from v_1 and v (the junction point of the Y-subgraph), apply homotopy (D). Thus, let us consider the case $b = v_1$ or $b = v$. Note that at least one of the vertices, a or c, has the property that the sum of its coordinates is smaller than the sum of the coordinates of $b = v$. Otherwise, the triangle (a, b, c) is not contained in the region XY. If $b = v$, replace the edges ab and bc with the edges ac and gb, where g is a or c, and has smaller sum of its coordinates than $b = v$. This creates a new junction point, g, and a longer path Γ_{x+y} with one less triangle or area.

Now, if $b = v_1$, then the change will be more drastic: delete the path Γ_{x+y}, replace the edges ab and bc with the edges ac and gb (where g is a or c, and has smaller sum of its coordinates than $b = v_1$), and, starting with the edge gv_1, build a pseudo-diagonal path, Γ'_{x+y}, that reaches the $x+y$-axis. Let the coordinates of g be (s,t). We have a new Y-subgraph, of smaller area, which is the union three paths, that converge at point g:

1. The pseudo-diagonal path Γ'_{x+y}, starting at g, obtained by construction. The important point v_1 is on Γ'_{x+y}.

2. Previously, v_1 was on Γ_x. Hence, edges following vertex c along Γ_x are pseudo-vertical. The edge ac is also pseudo-vertical. Thus the path that starts at g and continues with the old edges of the original Y-subgraph is a pseudo-vertical going down from g to the x-axis.

3. By assumption, the triangle (a,b,c) is where the sequence tilted away from the y-axis ended. Hence all edges previous to a must be pseudo-horizontal, as any vertical edge appearing before (a,b,c) is a place where a sequence of triangles tilted away from the y-axis must stop. As a consequence the pseudo-horizontal path that reaches g joins it to the y-axis.

Now, consider situation (ii) of Figure 5. The last triangle of the sequence of triangles tilted away from the y-axis is (a,b,c). The vertices (a,b,d) belong to the current Y-subgraph. Note that the edge ab is a pseudo-horizontal edge. Otherwise, if ab is vertical (x-coordinates remain the same), the sequence of triangles tilted away from the y-axis should have ended before, not at (a,b,c). Now if b is not equal to v, the junction point, or equal to v_1 we can simply apply homotopy (B). Hence, either $b = v$ or $b = v_1$. Note that we can assume that v and v_1 are different because the case when they are equal was solved when we discussed the case of v_1 lying along Γ_{x+y}. In this way v_1 comes after v going from the y-axis to the x-axis along $\Gamma_y \cup \Gamma_x$. Note also that the edge bd is a pseudo-vertical edge since v_1, v are in Γ_x. We claim that the vertex c has smaller or equal y-coordinate than d: else, the edge bc goes

down at least two levels and we can apply homotopy (C) to the triangle (b, c, d). Also notice that the x-coordinate of c is smaller than the x-coordinate of d because the triangle (a, b, c) is the last triangle tilted away from the y-axis.

Assume that $b = v_1$. Starting at the x-axis we can construct a sequence of triangles tilted away from the x-axis that follows along the path Γ_x. It ends, at the latest when reaching the edge cd. If it ends before reaching that particular edge, we can find a reduction; else, the sequence of triangles tilted away from the x-axis reaches cd, creating a triangle (c, d, e). Then simply apply a reduction of type (B) to the pair of triangles (b, c, e) and (c, d, e).

Finally, assume that $b = v$. Now we are going to construct a new Y-subgraph, containing v_2 in an XY region of smaller area: Notice that the edges ac and cd and all edges preceding them are pseudo-horizontal from the observations in the preceding paragraph. Take d as the starting point of a pseudo-diagonal path toward the $x + y$-axis, and use the pseudo-vertical edges that follow from d to get a path toward the x-axis. This pseudo-vertical path contains v_1. Clearly this is a new Y-subgraph where now the triangles (a, b, c) and (b, c, d) are missing.

We have completed the proof of the lemma. □

3 Complex orientation of T-curves

What is the connection between the real scheme $\mathbb{R}A$ of a curve A and the embeddings of $\mathbb{R}A$ in the set $\mathbb{C}A$ of complex points of A? It is well-known that $\mathbb{C}A$ is an oriented smooth connected compact two-dimensional submanifold of $\mathbb{CP}^2$. The curve $\mathbb{R}A$ is fixed set of points of $\mathbb{C}A$ under conjugation. The real curve may or may not divide $\mathbb{C}A$ into two parts. If it divides, we say A is of *type I*,; otherwise we say that A is of *type II*.

3.1 An orientation algorithm

We say that the real scheme of a curve is of type I (type II) if every curve of degree d with this real scheme is of type I (type II). When there are curves of both types

with the same real scheme, we say the scheme is of *indefinite type*. The classification of curves of indefinite type up to degree 6 was presented in [18] and also in [15]. In this section, we extend this classification for degrees 7 and 8.

Since T-curves come from a combinatorial construction, it is not surprising that the type of a T-curve can also be determined via a combinatorial process. Note that for a curve of type I, the natural orientations of the two components of $\mathbb{C}A$ define orientations on their boundary $\mathbb{R}A$. A curve is of type II if and only if it is impossible to give a consistent orientation of $\mathbb{R}A$ from an orientation of $\mathbb{C}A$. We are going to describe the combinatorial procedure, discovered by Itenberg and Viro, that defines the orientation of the real curve (when it exists) with respect to its complex Riemann surface. We will describe it only for primitive triangulations. We need some preliminary observations about the triangles in the T-curve. The following lemma is valid for triangulations of $\mathbb{Z}^n$, although we state it for the concrete case of $n = 2$.

Lemma 3.1 *Let $\mathcal{T}$ be a* primitive *degree d triangulation of $P(T)$, and let (v_1, v_2, v_3) denote a non-degenerate simplex in $\mathcal{T}$. Let $\sigma : \mathbb{Z}^3 \to \mathbb{Z}_2^3$ be the "parity function" whose ith component $\sigma_i(v_1, v2, v_3)$ is 0 if v_i is even, and is 1 if v_i is odd. Then the vectors $\sigma(v_i)$ are mutually distinct.*

Proof: Suppose not. Then without loss of generality, $\sigma(v_1) = \sigma(v_2)$. Consider the parametrization of the edge between v_1 and v_2: $s(t) = v_1 + (v_2 - v_1)t$, $t \in [0, 1]$. Note $\sigma(v_2 - v_1)$ is the zero vector, since each coordinate is even, so $s(1/2)$ is an integer vector. This is impossible, since the edge of a primitive triangulation cannot intersect another vertex of the triangulation. □

Lemma 3.1 restricts the positions of triangles in a primitive triangulation. More importantly for us, in a two-dimensional primitive triangulation, if both coordinates of v_1 are even integers, and both coordinates of v_2 are odd, then v_3 must have one coordinate that is even and another that is odd.

Lemma 3.2 *Let $\mathcal{T}$ be a two-dimensional primitive degree d triangulation of $\mathbb{RP}^2$. Let $T_1 \in \mathcal{T}$ be a triangle in the first quadrant with integer vertices v_1, v_2, v_3, and let*

$T_i, i = 2..4$ be the other triangles in the orbit of T_1 under the reflection group. Then under Viro's patchworking construction,

1. *there is one T_j that does not contain a piece of the T-curve,*
2. *for $i = 1..3$, there is a triangle that contains a T-curve opposite to a vertex in the orbit of v_i.*

In particular, if a triangle contains an edge on a coordinate axis or an edge on infinity, then the orbit of this triangle intersects at most two distinct components of the T-curve. If a triangle contains two edges that lie on a coordinate axis or on infinity, then the orbit of this triangle intersects at most one distinct component.

Proof: By the Lemma 3.1, we may assume that the coordinates of v_1 are even, and the coordinates of v_2 are odd. Then by the rules of changing signs and by changing "+' to "−" if necessary, we may use the reflection symmetries to assume that $\mathrm{sgn}(T_1) = [+++]$, and thus T_1 does not intersect the T-curve. Then the sign rules imply that $\mathrm{sgn}(T_3) = [++-]$ and either $\mathrm{sgn}(T_2) = [+--]$ and $\mathrm{sgn}(T_4) = [+-+]$, or the signs for T_2 and T_4 are switched. In either case, parts (1) and (2) of the lemma are proved, since the T-curve will cut across T_3 opposite from an image of v_3, and will cut across T_2 and T_4 opposite from images of v_1 and v_2, as illustrated in part (i) of Figure 6.

Figure 6(ii) illustrates the case in which a triangle contains a special edge on an axis of reflection. The key observation is that one of the triangles can not intersect the T-curve. That forces the other triangle that shares the special edge to intersect the T-curve parallel to that edge. Thus the two remaining triangles must contain a portion of a T-curve that intersects the orbit of the special edge. Repeating this argument for the case of a triangle with two special edges (as illustrated in the triangulation of Figure 6(iii)) completes the proof. □

Orientation of T-curves: As we just saw, every triangle (v_1, v_2, v_3) of the patchworking process has four copies. The T-curve cuts through the triangle in question in three of the copies and leaves one of the copies empty. Each edge v_iv_j has four

copies but only two are touched by the T-curve. Identify the two empty copies of edge v_iv_j, preserving the orientation of the triangle. Also identify the two copies of v_i, v_j that are touched by the T-curve. If we do this for the three edges of triangle (v_1, v_2, v_3), we get a simplicial complex with four triangles, isomorphic to a projective plane. The three edges of the T-curve inside the copies of (v_1, v_2, v_3) become identified into a single component (a pseudoline). Clearly, the orientation of the other two edges is now given by what was the orientation in one of them. For example, in Figure 6(i), if the T-curve segment in the second quadrant is oriented towards edge $v_2'v_3'$, then it induces an orientation in the third quadrant towards edge $v_1''v_3''$ and in the fourth quadrant towards edge $v_1'''v_2'''$.

The initial step in the orientation algorithm is to choose any triangle, T_1, of the triangulation, and fix an orientation for a segment of the T-curve that is contained in one of the copies of T_1. That simultaneously determines the orientation of at least one component of the T-curve, and of the other two copies of the segment. One of the oriented segments points towards a triangle, T_2, that contains a T-curve segment that has not yet been oriented. Let the T-curve segment contained in T_2 inherit the orientation from T_1, and then orient the copies of T_2 as above. The algorithm successfully completes when all segments of the T-curve are oriented, or fails to complete when we attempt to give an orientation to a segment that has already received the opposite orientation.

Theorem 3.3 ***(Itenberg-Viro)*** *The T-curve associated with a given regular triangulation and signs for its vertices is of type I if and only if the orientation algorithm terminates with a unique orientation for each edge of the T-curve. If the orientation is not consistent, i.e. an edge received two orientations, then the curve is of type II.*

3.2 New results on complex orientations of T-curves

In this section we present three new contributions to the study of complex orientations of real curves of degrees 7 and 8:

- A list of schemes that are realizable as T-curves (presented in Figures 7–10 and Tables 7–8).
- Certificates of indefinite schemes (Tables 1–8).
- A comparison of T-curves that are type I according to Theorem 3.3, but come from nonregular triangulations, with known properties of algebraic curves of type I.

We begin by presenting in Figures 7–10 and Tables 7–8 the results of our efforts to construct T-curves in degrees 7 and 8. Recall that an M-curve is one that contains $M = (d-1)(d-2)/2 + 1$ components, and that an $M - s$ curve contains $M - s$ components. Note that most of the T-curves we have found are $M - s$ curves with relatively small values of s, in contrast to previous tabulations of realizable schemes in which the constructions have focused on $M - s$ curves for small values of s. Our experience so far indicates that this is a general characteristic of random T-curves, as the symmetry of the construction tends to limit the number of ovals that are interior to another oval.

We classify each scheme using a variant of Viro's notation (see [22]). J denotes a pseudoline. $\langle\alpha\rangle$ denotes α unnested ovals, whereas $\langle 1\langle\beta\rangle\rangle$ denotes a single oval surrounding β unnested ovals. The union of these two schemes is indicated by the juxtaposition of their symbols, as in $J\ \langle\alpha\rangle\ \langle 1\langle\beta\rangle\rangle$. The cartoon depiction of this scheme is provided in Figure 7.

The figures show the schemes that we were able to realize with T-curves, and also indicate which schemes we have determined to be indefinite. Boxes, triangles, and circles indicate schemes that we constructed using patchworking. Boxes are proven to be of type I, whereas triangles indicate schemes of type II. Filled circles represent schemes of indefinite type, and are our new contribution to this classification. Open circles represent schemes for which we have partial information (e.g., we have an example of a non-dividing curve for this scheme). The figures also show diagonal lines. If a scheme exists along these lines, then it is of definite type.

Note that the consequences of known theorems about dividing curves are visually apparent in the figures. In particular, Arnold's congruence implies that indefinite schemes in degree 8 must lie along certain diagonals in the figures.

The obtainable schemes of degree 8 are not yet classified, and no one yet knows which schemes are obtainable as T-curves. Thus the data presented in this paper also gives concrete data to support the questions, "which schemes exist in degree 8?", "which of these are constructible using Viro's method?", and "which of these are indefinite?"

Theorem 3.4 *Each real scheme indicated as indefinite type in Tables 1–8 is indefinite and is obtainable as a T-curve.*

Proof: Tables 1–8 give the proof that these schemes are indefinite. Each entry gives the sign vector and the regular triangulation used to produce each scheme. The regularity of each triangulation is proved in Appendix 1. The sign vectors are given in a column-dominant manner, beginning from the origin and moving up each column. Thus the vector $+--+++---+--+----+++++++----+--+----$ is interpreted as the vector for degree 7 that assigns the vector $+--+++--$ to the vertices $(0,0)\ldots(0,7)$, while the vectors $-+--+--$, $--++++$, $+++--$, $--+-$, $-+-$, $--$, and $-$ are assigned to the other columns. □

Only a handful of general theorems are known for dividing curves. We present several of the general theorems here; see [22] for a complete list.

Theorem 3.5 *The following properties may be used to determine whether a scheme is type I and/or type II:*

1. *M-curves always divide.*

2. *$M-s$ curves never divide when s is odd.*

3. *If ℓ is the number of ovals in a curve of type I, then $\ell \equiv \lfloor d/2 \rfloor \bmod 2$.*

4. ***Arnold's congruence:*** *If d is even and a curve is type I, then $p-n \equiv d^2/4 \bmod 4$, where p denotes the number of positive ovals (ovals nested within*

an even number of other ovals) and n is the number of negative ovals (the remaining).

5. ***Rokhlin and Rokhlin-Mishachev formulas:*** *If d is even and the curve is type I, then*

$$2(\Pi^+ - \Pi^-) = \ell - d^2/4.$$

If d is odd and the curve is type I, then

$$\Lambda^+ - \Lambda^- + 2(\Pi^+ - \Pi^-) = \ell - (d^2 - 1)/4.$$

In the preceding formulas, Π^+ is the number of pairs of nested ovals whose orientations are induced by an orientation of the annulus they bound. Λ^+ is the number of ovals that bound a disk D and whose integral homology class in $H_1(\mathbb{RP}^2 \setminus D, \mathbb{Z})$ has opposite sign from the homology class of the pseudoline. Π^- and Λ^- are the number of remaining nested pairs and ovals, respectively.

The data for regular triangulations indicates that Rohklin's comment twenty years ago [18] continues to hold in light of this new data: nothing so far contradicts the conjecture that all real schemes not prohibited by known theorems are of indefinite type.

As discussed in the first part of this paper, it is natural to think of the construction of T-curves as a purely combinatorial procedure, valid for any triangulation. Similarly, the algorithm for orienting a T-curve may be applied to T-curves coming from nonregular triangulations. We have applied this algorithm to millions of T-curves in degrees 7 and 8, finding thousands of curves that are type I according to Theorem 3.3. *None of the curves that could be oriented violated any of the results that are known to be true for real algebraic curves.* This was somewhat surprising, since we do not know of a way to associate any sort of a complex object to a T-curve that comes from a nonregular triangulation. On the basis of this experimental evidence, we reiterate the question asked at the beginning of this paper: Do combinatorial T-curves constructed from non-regular triangulations, still yield schemes of real smooth algebraic curves?

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A Proofs of Regularity

In this appendix we present the proofs that the triangulations used in Section 3. These proofs are presented in Figures 11–13.

Figure 11 contains pictures of two triangulations that are members of the family of placing triangulations (on the left) and two that are perturbations of the Delaunay subdivision (on the right). These families provide well-known regular triangulations [14].

For the triangulations in figures 12 and 13, we explicitly present a proof of regularity by exhibiting a piecewise-linear convex function on the vertices. The number at each vertex may be interpreted as a height. The reader may verify that the lifting of any two adjacent triangles is concave up.

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α	β	Tri.	Sign Vector
0	4	7D	D: + + + + + − − − + − + + − + + − − + + + + − + − − + + + + + − + − + −
		7D	N: + + + + + − − + + + − + + − + + − + + − − + + + − − − − − + − + + − − − + − +
1	5	7D	D: − − + + − + − + + − − − − − + − − − + − + + + − − − − − + − − − + + − +
		7D	N: + − − − − − − − + − − + − + + + + + + + + + − − − + − − + − + + − − + − +
2	2	7D	D: + + + + + − − − − + + − − − − − + + − + − + + + − − + + − + − − − − −
		7D	N: + + + + + + + + + + + + + − − + − − − + − + − + + − − − − − − − − + + − −
2	4	7D	D: + + + + + + + − + − − − − − + − − + − + − + − + − − − − − + + − − + + + + +
		7D	N: + + + + + + + − + + + − − − + + − − + + − + − + − + − − − + − − − − − − −
3	1	7D	D: + − + − − + + − + − − − − + − − + − − − − + − − − − − + + − + − − − + + +
		7D	N: − + − − + − + + − − − − + + − + + − + + + + − − − − + + + − − − + + − +
3	3	7D	D: + − + − + − + + + − + − − − + − + + + − + + − + − − + − + + + − + + − −
		7D	N: + + + + + + + + − − + − + − + − − − − + − + − + − + + − + − − + − + + + + −
3	5	7H2	D: + + + + − − + − + − + − − + + + + − + − − + − + + − + − − − − − − − − +
		7H2	N: − − − − − + − + + + + − − − − + − + + − + + − + + − − + + + + − − + + +
5	1	7D	D: + + + + + − + − − + + + − − − − + − − + − − + + + − + + − + − + + + + − +
		7D	N: + + + + + + + + + + + + + + − − − + + − + + + + + + + − + − − − + + + −
5	3	7H2	D: + − − + + + − − − + − − + − − − − + + + + + + + − − − − + − − + − − + − − − −
		7H2	N: + + + + − + + − − + + + − + + + + − − − + − + − − + + − − + − + + + + − −
5	5	7H2	D: + + − + − − + + + + + − − + + − + − + − + − − + − + + − + + − − − + − + + −
		7H2	N: + − + + + − − + + + + − − + − − + + + − − − − − − + + − + + + − − − + − + +
6	2	7D	D: + − + − + − + + + − + − − − + + + + − − + + − + − − + + + + + − + + − +
		7D	N: + + + + + − + − + + − + − − − + + + + + − − − + + − − + − + − + − − − − + −
6	4	7H2	D: + + + + − − + − − + + + + + + + − − + + − − + − − + − − − − − − + − − + +
		7H2	N: + − + + + − − − + + − − + − + − + + − − − − − − + + − − + − − + − + + −

Table 1: Degree 7 triangulations and sign vectors that produce T-curves with scheme $J\langle\alpha\rangle\,\langle 1\langle\beta\rangle\rangle$ and are type I ("D") and type II ("N").

α	β	Tri.	Sign Vector
7	0	7D	D: + − + + + + − + − + − + + + − − + − + + + − − + − + − + + + + + + + +
		7D	N: + + + + + + + + + + + + − − − − + + − + − + − + − + + + − + + + − − − − − +
7	1	7D	D: + + + + + + + + + − + + + + + + + − + − + + + + + + + + + − − − − + − +
		7D	N: + + + + + + + + + + + − + − + + − + + + + − + − + − − + + + − + − + − +
7	3	7H2	D: + + + − − − − − − + + − − + + − − − − − + − − − − − + + + + − + + − + − + − +
		7H2	N: + + + + + + − + + − + + − + + − − + + + + + + + − − − + + − + + − + − + − +
8	2	7H2	D: + + + + + + − − + − + − − + + − + + + + + + − − − − − + − + + + + − − − − −
		7H2	N: + + + − + − + + − + − + − + − − + − + − + − + − + − − + + + + + − + − + − +
8	4	7H2	D: + − − − − − − + − + − − + − + − − − − − − − − − + + − + − + + + + + − +
		7C	N: + + + + + + + + − − − + + + + + + + − + − − + + + + − + + − − − + − − + +
9	0	7D	D: + + + + − + − − − + − + − + + + + + − + + + − + − + + − + + − + + + − + − − −
		7D	N: + + + + + + + + + − + − + + + − − + + + − − − + − + + − + + + − − − + − + −
9	1	7D	D: + − − − + − − − − − + − + − + − − − − − − − − + − + + + − − + + − + + −
		7D	N: + + + + − + + − − + + − + − + + + + + + + − + + − + + + + + − + + − − +
9	3	7H2	D: + + + + − + − − − + − − − − − + + − + − + + − + + − − − − − + − + + − +
		7H2	N: − − + + + − − + + + − − + − − + + + + + + − − − + + − − + − + − − − − − −
10	2	7H2	D: + + + − + − − − − + + + + + − + − + − + − + − + + − − + + + + + + − − +
		7H2	N: − + + − + + − − − + + − − + + − + − + − + − + − + − − + − + + + + + + − +
11	0	7D	D: + + + + + + − − + + + − + − + + + + + + + + − + − + + − + + + − + + + +
		7D	N: + + + + + + + + + + + + + − − − − + − + − + − − + + + − − − + + + + − + +
11	1	7H4	D: + + + + − + + − − + − + + − + + + + + + − − + + − − + − − − − − − + − −
		7H4	N: + + + − − − − + − − + − − + − + − + − − + + − + − + + − + − + − − − + − + − −
13	0	7C	D: + + + + + + + + + + + + + + + + − + − + + + − + + + + + − − + − + − + + + − +
		7C	N: + + + + + + + + + + + + + + + + − + − + − + + + + + + − − − + − + − + + +

Table 2: Continuation of degree 7 triangulations and sign vectors that produce T-curves with indefinite scheme $J\langle\alpha\rangle\,\langle 1\langle\beta\rangle\rangle$. The scheme $J\langle\alpha\rangle\,\langle 1\langle 0\rangle\rangle$ is equivalent to the scheme $J\langle\alpha+1\rangle$.

α	β	Tri.	Sign Vector
0	9	8D	D: + + + + + + + + + + + - - + + + + + - + - - - + - + + + - + - - + - - - - - - + - - - - - + -
		8D	N: + + - + - + - - - + - - - - - - + + + - - - + - + - - + + - - - + - + - + + - - - - + - +
1	6	8D	D: + + + - + - + - + + + - + - - - + - - - - - - + - + + + + + - - - - + - - + + + + - + - + +
		8D	N: + + + + + + + + + + + + + + - - - + - - - - + - + + - + + - - - - - + - - - + - - - + + -
1	10	8D	D: + + + + + - + + + - + + - + - - - - + - + - - - - + + - - - + - - - + + + - - + - + - +
		8D	N: + + + - + - - - - - - - + - - + - + - + - - - + + - + - + - + - + - - - - - + + - + - + - - - +
2	7	8D	D: + + + + + + + + - - - - - + - - - - + - + - - - + - + - + - + - - + - + - - - - - + - + - + - - +
		8D	N: + + + + + - + + + - + - - + + - - - + - + + - - + + + - + - + - + - + - - - - - - + - + - +
2	11	8D	D: + + - + - + - - - - + - + - + - + + - - - - + - - + - + - + + + - - - + - + + + - + + -
		8D	N: + + + + - - + - + + + - + - + - + - - - + - + + + + - + + + - - - - - - + + - + - - - + + -
3	4	8D	D: + + + + + - - + + + - - + + - + - - - - + - + + - - + + + - + - + - + - - + + - + - - + +
		8D	N: + + + + + + + + + - + + - + + + + - + - - + - + + + - - - - - - - + + - - + + + -
3	8	8D	D: + + + + - + - + + + + + + + + + + + + + - + - + + - + - + - + - - + - - + - - + - - + +
		8D	N: + + + + - + + + - - + - + + - + + + + + + + + + - + + + - - - + - - + - - + - - - + + -
4	5	8D	D: + + + + + + + + + - + + + - + - + - - - + - - - - + + + + + + + - + + + + + - + - + - +
		8D	N: + + + + + + + - + + - + - - - + - - - - - - - + - + - + - - - - + + + - - - + - + + + + + +
4	9	8D	D: + + + + - + - - - + - - - + - - - + - + - - + - + - + - - - - - - + - + + + - - - - - - + +
		8D	N: + + + + + + + - - + - - - - + - - - - - - - - + - - + - - - - - - - + - - + - - + - + - -
5	6	8D	D: + + + + + + - - + - - - + - + + - + - + + + - - - - + + + - + - - - - - - + - - - + - +
		8D	N: + + + + - - - - - - - - + - - + - + + + + + - - - - - - - - - - + - - + + + - - - - + + - - -
5	10	8B	D: + + + + - - - + - + - + - + + + + + + - - - + - + - - - - - + + - - - + - - + - + + -
		8B	N: + + + + + - + + - - - + - - + + - + - + + + + + - + - + + - - - + - + + + - + + - - - +
6	3	8D	D: + + + + + + + - - + + + + - + + + - - - + - + - + + - + + + + + + + + - + - + - + + + +
		8D	N: + + + + + + - + + + - + + - + + - - - - + - - + - + - + + + + + + + + + + - + + - + - - -
6	7	8D	D: + - + + - + - - - - - - - - - - - + + - + - - + - + - + - + - + - + + - + - + - - - - - + -
		8D	N: + + + + + + + + - - - - + - + + + - - - - + - + - - + - + - - - - - - + + - + - - - + - + - -

Table 3: Degree 8 triangulations and sign vectors that produce T-curves with indefinite scheme $\langle\alpha\rangle\langle 1\langle\beta\rangle\rangle$. The scheme $\langle\alpha\rangle\langle 1\langle 0\rangle\rangle$ is equivalent to the scheme $\langle\alpha+1\rangle$.

α	β	Tri.	Sign Vector
7	4	8D	D: + + + + + + + + + + + - + + + - - + + - + + + - + - - + + + - - + - + - - - + + - - - + - +
		8D	N: + + + + + + - - + - + - + + + - - + - + - + - - - + + + + + - + - - + - - - - - - - - - +
7	8	8B	D: + + - + - - + + - + + + + + + + + - + - + + + + + - + + + - - - - - + + + - + + + - -
		8B	N: + - + - - + - + - - - - - - + - + - - - + - - - - + - - - - + - - + - + - + + + - + - - - + +
8	5	8D	D: + + + + + + - + + + - - - - - - + + + - + - + - - + - + - - - - - - - - - + - + - + - + - - -
		8D	N: + + + + + + - - + + - - - - - - + + + + - + - - - - - - - - - - - - - + - - + + - - - + - +
9	2	8D	D: + + + + + + + + - - + + - + - - - + + + + + - + + - - + + + - + - - - - - - - + + - + - - -
		8D	N: + + + + + + + + + - + + + + - + - + - - + + - + + + - + + - + + + + + - + - + - - - - + +
9	6	8D	D: + - + + + - + + + - - - - - - - + + + - + + + - + - - + - + - - - - - - + - + - - + - - - +
		8D	N: + + + + + - + + + - + - + + - + + + + + + + - + - + - + - + - + - - - - - + + - + - - - - - -
10	3	8D	D: + + + + + - + - + - + + + + + + - + - + - + + + - + - + - - - - - + + - + + + + - + -
		8D	N: + + + + + + + - - - + - + + - + + + - + - + + + + - + - - - - + - + + - - - + - - - - - + -
10	7	8C	D: + + + + + + + + - + + + - + - - + - + - + - + - + - + + - - + - + - + - - - + + - + - - +
		8C	N: + + + + + + - + + + - - - + - - + + - + - + - + + + - - - - - + - + - - - + - - - - - - +
11	4	8C	D: + + + + + + + + - + + + - - + - - - - - - - - - + - + + - - + + + + + + - + + - - - + + +
		8D	N: + + + + + + - + + - + - + - + - + + + + + - - + - + + + + + + - + - + - + + - - -
12	1	8D	D: + + + + + + + + - + + + - + + + + - + + + + + - - - + - - + + + + + + + + + - - + - + + +
		8D	N: + + + + + + + + + + + + + + + + - - + - + + + + + - - + + - + - + + + + + + - - + - + - +
13	2	8D	D: + + + + + + + - + - + + + - + + + + - + - + - + + + + + - + + + + - + - + + + - - - - + +
		8D	N: + + + + + - - + - + - - + - + + - + + + + + + - + - - + - + + + + + + + + - + + - + - - -
14	3	8D	D: + + + + + + - + + - + + + - + + + + - + - + - - - + + + + + + - + - + + + + + + + + - -
		8D	N: - + - + - - - - + + + - + + + + - - - - - - + - - + - + + - - - - - - - + - + + + + - + +
15	0	8D	D: + + + + + + + + + + + - + - - + + + + - + - + - - + - - + + - + + + + + - + + - + + + +
		8D	N: + + + + + + + + + + + - + + + - - + + + + + + - + + - - - + + + + + - + - + + + + + + + +
16	1	8D	D: + + + - + + + + + + + + + - - + - + + + + + + + + - - - + + - + - + + - - - + - + + - +
		8D	N: + - + + + + + + - - + + + - + - - + - + - + + + - - + - + + + + + + + - - + - + + + + + -
17	2	8D	D: + - + + + + + - - - - + - + + - - + + + + + + + - + - + + + + + + + + - + - + + + - + -
		8C	N: + + + + + + + + + - - - + + - + - + + + + + + + + + - + + - + - + - - - + + + + + - -

Table 4: Degree 8 triangulations and sign vectors that produce T-curves with indefinite scheme $\langle\alpha\rangle\,\langle 1\langle\beta\rangle\rangle$ (continued).

α	β	Tri.	Sign Vector
1	10	8D	D: ++−−−−−++++−+−++−++−+−−−++++++++−+−−++++−−+++
		8D	N: −+−−+−+++−+++−++++++−+++−++++−++−+−++++−++−
2	3	8D	D: ++++++++++−++++−−−−−−−−−++−+−++++++−−+−−+−+−−
		8D	N: +++++++−++−+−−−−++−+−−−−++−−−−+++++++−−+−−+++
2	7	8D	D: −++−++−+−+−−−−−−+−++++−−−+−+−−+++++−+−++−−++
		8D	N: −++++++++−−−++−+−−+−−+−−++++++++−−−+−−+−+−−−+−+
4	5	8D	D: −−−−++++−++−+−+−−−+−++++−++++−+−++−−−−+−+++−
		8D	N: ++++−+−−+++−−+−−+−++−−−−−+−−−−++++−−+−−+−+−−+
5	2	8D	D: +++++−−−−+−−++−++++++++−−−−++−−−−−−+−−++++−+−
		8D	N: +++++++−+−−+−+−−−−+−++++−++++−+−−+−−+++−+++−−
5	6	8B	D: +++−+−−+++−+−−−++−++++−−−++++−+−+−−−−−++−+++
		8B	N: +++−−−−−++++−++−−−++−+−−++−+−++++++−++−+−−−
6	3	8D	D: +++−−++−−+−+−−−+−+++−−−+−−+−+−+−++−++−+−−−+−+
		8D	N: +++−+−−+−−+−++−−++−+++−−+−−−++−−−−+−−++−+−−+−
7	4	8B	D: +−+++−−−+++−+−−++−−+−+−−−++++−−++−−++++++−++−
		8B	N: +++−−−−+−+−++++−++++−+−+−++−++−++++++−−++−++
8	1	8D	D: ++++++−+++−−++−−++++++−−−−−++−++++++−−+−−−+−−
		8D	N: ++++++−+−−+++−−−+−−+−+−−++++−++−−−++++++−+−++
9	2	8D	D: ++++++−−++−+−+−+++−+++−−−+−+−++++++−+−−−−−−
		8D	N: ++++++−−+−+−−−−++++−−+−−++++−+−+−++++++−+−−+
12	1	8D	D: +−+++−−+++++−−+−+−+−+−−−+−++−+++++++−++++−+
		8D	N: +++−+−−+−−++−++++++++−++−+−−+++++−+−++−++++
13	2	8B	D: +++−++−−−+−+−++++−++++−−−+−++++++++−+−+++−+−
		8C	N: ++++++++++++−−−+−+−+−+−+−−+−−−++−+++−−−−−−+++
16	1	8D	D: −−−−−−−−+−−−−+−++−+−−−−−+−−−+−−+−+−+−−−−+−−−+
		8C	N: +++++++++−−−++−+++−++−++++−−+−+−+−+−++++++−−+

Table 5: Degree 8 triangulations and sign vectors that produce T-curves with indefinite scheme $\langle\alpha\rangle\,\langle 1\langle 1\rangle\rangle\,\langle 1\langle\beta\rangle\rangle$. Curves with $\beta = 0$ are reported with the schemes $\langle\alpha\rangle\,\langle 1\langle 1\rangle\rangle$.

α	β	Tri.	Sign Vector
0	5	8D	D: + + + + + + + + - + - - + + + - - - + - - - - - - - + + - + - - + - - - - - - + + + + - + -
		8D	N: + + + + + + + - + + - - + - - + - - - - + - - - + - - + + + - + - - - - + + - + + + + - +
0	9	8D	D: + + + + + + - + + - + + + - - - + - + - + - - + - - + - + + - - - - - + - + - - + - - + -
		8D	N: + + + + - + + + - - - + - - - - - - - - - - - - - + - - - + - - + - + - + - - - - - + + + +
1	2	8D	D: + + + + - + + - + + + + + + - - + - - - - + - - + + + + + - + - + - - - - - - - - - - +
		8D	N: + + + + + + - + - + + + - + - - - + - + - + + - + - + - + - - - - + - - - - - + + + + - - + - + +
1	6	8D	D: + + + + + + + - + + - - + - + + + - - - + - - - + - + + + - - - + - - + - - + - + - + - - -
		8D	N: + + - + - + - - - - - + + - - + + - - + + + + - - + - - + - - + - + + + + + + - - - - + -
1	10	8D	D: + - - + - + - - - - + + + - + + + + + - - - - + + + + - + + - - + - - + + + - + + + - -
		8D	N: + - - - - - + - - - - + - - - + - - + - - - - + - - - - - + - + - + + + - - - - - - + + +
2	3	8D	D: + + + + - + + - - + + + + + - + - - - + + - - + - + - - + + + + - - + - + - + - - + -
		8D	N: + + + + - - - - - + + - + + - - + - + - - + + + - + - + - - + - - - - - - - + + - + + + -
2	7	8B	D: + + + - - - + - + - - - + - + - - - + - + - + + + + - + + - - - - + - - - + - + - + + -
		8B	N: + + + - + + + - - - + - - + + - - + + - - - - + + - + - + - + + - - - + - + - + - - + + + - -
2	11	8C	D: + + + + + + + - + + + - + + + + - - - - - - - - + + + - - + - + - + - + + + - - - - - - +
		8C	N: + + + + + + + + - + - - + + + - + - + - + - + - + - + + + + - + + + + + + - - + + - + - + +
3	4	8D	D: + + + + - - + + - - - - - - + + - + - + + - + + + - + - + + - + - - + + + - + - + - + - +
		8D	N: - + + + - + - + - - - + - - - - - - + + - + - + + + + - + - + - + - + - + + + - + - + + - +
4	1	8D	D: + + + + + + - + + + + + - - - + + + + + + + - - + + + - + - - + - - - + - - + + - + - + - - -
		8D	N: + + + + + + + - + + + - + - - - - - + - - + - + - + + + - - + + + - - - - - + - + + - -
4	5	8B	D: + + - + + - - + + + + + + + - - + - + - + + + - - + + + - - + + + - - - - - + + - + - - +
		8B	N: + + + + + - + + - + + + - - - + - - + + + + - + + + + - - - + - + - + + - + - - - + + - -
5	2	8D	D: + + + + - + + - + - - + - - + - - + + + - - + + - - - - - + - - - + + + - - - + - + - + -
		8D	N: + + - + - - + - + + + + + + - - + + + - - - + - + + + + - - + + - - - + + + + + + + + +
6	3	8B	D: + + + + - + + + - - - + - - + + - - - + - + + + + - + + - + + + + - + - - + + + + + - -
		8B	N: + + + + - + - + + - + - + - - - + - + - + - - + + + - - + + + + + - + + - - + - + + + +
7	0	8D	D: + + + + + + + + + - + - - + - - - + - + - + + + + + - + - - - - - - + - - + + - + + + -
		8D	N: + + + - - - - - - + - - - + + - + - + + + + + + - - - + + + - + - - + + - - + + + + - - -
8	1	8B	D: + + + - - + + - - - + + + - - - - - + + + - + + + + - - + - + - + - + + + + + - + + + -
		8B	N: + + - + + + - - - - + + - + - - + - + + + + - - - + + + - - - + - + + + - - + + - + - - - -
9	2	8B	D: + - + - - - + + - - - + - + - + - + + - - + - + - - + + + - - + + + - + - + + + - - - - + + + - -
		8D	N: + + + - - + - + + - + + + - + - + + - + + + + + + + - + - + - - + + + + - + - - + - - -

Table 6: Degree 8 triangulations and sign vectors that produce T-curves with indefinite scheme $\langle\alpha\rangle\,\langle 1\langle\langle\beta\rangle\ \langle 1\langle 1\rangle\rangle\rangle\rangle$.

α	β	γ	Tri.	Sign Vector
0	0	4	8D	D: + + + + + + + + - + + - + + - - - - - - + - - - + - - - - + + + + - + + + + + + - + - - -
			8D	N: + + - + - + - + + + - - + - - + + + - - + - + - + - + - - - + + + - + + + + + + - + + - - + + +
0	2	2	8D	D: + + + + + + - + + - + - + + - + - - - - + - - + + - - - + + + + + - + + - - - - - - + - +
			8D	N: + + + + + - - + - - + - + + - - + - + - - - + + + - + - + + - + - - + + - - + - + + + + + + -
0	3	3	8D	D: + + + + - + + + + + + - + + + + - - - - - + + - - - - - + - - + - + + - - - - + + + + + -
			8D	N: + - - + + - - + + + - + - - - - - + + - + + + + + - - - + - - - - - - + + + - + - - - - - - + +
0	6	2	8D	D: + + + + + + - - - + + + + + + - - + - - - - + + - - - + - - + - + - + - + - - - + + + - + +
			8D	N: - + + - + + + + - - - + - + - + - + + - + + - - + + - + - - + + + + + + - - + - - - - + + -
1	1	4	8D	D: + + - + - + + + + - - - + + + - + - - + - + + - + - - + + + - - + + + - - - + + + - - + +
			8B	N: + + + + - - - + - + + + - - - + + + - - - - + - - + - + + + - - - - + - + + + + - + - - +
1	3	2	8D	D: + + + + - + + - - + - + - - + + + - + - - + + - - - - - - + - + - + + - + + + + + + + - +
			8D	N: + + - + + + + - + + - - + + - - + + - - + + + + + + - + + - + + - - + + + + + - - - -
1	7	2	8D	D: - + + + + - - + + + - - - + - + + - - + + + - - - - - - - + - - - + - + + - - - + + - - + -
			8C	N: + + + + + + + + - - - - + + - - - + + - - - + + + - + + - + + - + - + + + + + - - - - + +
2	0	2	8D	D: + + + - - + + + - - + + + - - + + - + + + + - + - + + + - - + - + - + - - + - - + + + + - + +
			8D	N: + + - + + + + + + - - + - + + - + - - - + + - + + + - + + + + + - + + + - + - + - - - - -
2	1	3	8D	D: + + - - - - - + - - + + + + - - + - - - + + + - - - + - + - - - - + + + + - + - - + + - + +
			8B	N: + + + - - - - - - + - - - - - - - + + + + + + + - + - - + + - + + - + + - + - - - + - - - - - -
3	1	2	8D	D: + + + + + - - + + - + - + - + + + + + + + + - - + + - - - - - + - - - + + + + + + + + + - + +
			8D	N: - + + + + + - - - + - + - - - - + - + - - - - + + + + + + + - - - - + - - + + + + + + + + - +
4	2	2	8D	D: + + - + - + + + - + - - - + + + + - - + + + + - - - + - + + - + + + + + - + - - - + + + -
			8B	N: + + + + + - - + - - + - + - - + - - + + - - + - + - + - + - + - - - - - - - + + + + + + - - - -

Table 7: Degree 8 triangulations and sign vectors that produce T-curves with known indefinite schemes of type $\langle\alpha\rangle\,\langle 1\langle\langle\beta\rangle\ \langle 1\langle\gamma\rangle\rangle\rangle\rangle$, $\beta \neq 1, \gamma \neq 0$ and $\gamma \neq 1$.

Scheme	Tri.	Sign Vector
$\langle 1\rangle\,\langle 1\langle 2\rangle\rangle\,\langle 1\langle 5\rangle\rangle$	8D	D: + − + − + − − + + + + − + + − − + − + − − + − + − + − + + − − + + + + − +
	8D	N: − − − + − − + + + − − − − + − − − + − + + − − − − − − + + + + − + + + − +
$\langle 2\rangle\,\langle 2\langle 1\rangle\rangle\,\langle 1\langle 3\rangle\rangle$	8D	D: + + + + + − − + + + + − + − − + − + − − − + − + + + + − + − + − + + + − + − − + −
	8D	N: − + − − − + − − + − − − + + − + + + − + − + + + − − − + − + − − + − − + − − − − − −
$\langle 3\rangle\,\langle 1\langle 2\rangle\rangle\,\langle 1\langle 3\rangle\rangle$	8D	D: + + + − + + + − + − + − + − − − − − + − + − − + + − + + − + + + + + + + − + − + − − − +
	8D	N: + + + − − + + + − + − − − − − − + − + + − − − + + + + − + − − + + − − − + + + − + −
$\langle 4\rangle\,\langle 3\langle 1\rangle\rangle$	8D	D: + + + − + + + − + + − − + − − + − + − + − − − − − + + − − + + + − + − + − − + − + + − +
	8D	N: + − + − + − − + + − + + − − + − + − − − + − − + + + − − + + + + − + + − + − − −
$\langle 5\rangle\,\langle 2\langle 1\rangle\rangle\,\langle 1\langle 2\rangle\rangle$	8D	D: + + + + + − − + + − + − + + + − − + − + − + − − − + + − + − + − − + + + + + + + − + + − + − + +
	8D	N: + − + + + − − + − − + + + − − + + + − + − − + − + − − + + + − − + + + − + − − − − + − + + +
$\langle 14\rangle\,\langle 2\langle 2\rangle\rangle$	8C	D: + + + + + + + + + + − − − + + + + + − + − + − + + + + − − − + − + − + − + − − − − − + − + − +
	8C	N: + + + + + + + + + + − + − − + + + + − + − + − + + − − − + − + − + − + − − − − − + − − − +

Table 8: Degree 8 triangulations and sign vectors that produce T-curves with indefinite scheme of miscellaneous type.

Figure 1: Examples of Y-subgraphs for d=7. The first image presents an example of a pseudo-horizontal path, Γ_y, that is not pseudo-diagonal; it fails precisely at the edge before v. Both images show dashed lines represent three directions—horizontal, vertical, and diagonal—that underlie every triangulation. The actual triangulation for the Y-subgraphs is not shown.

Figure 2: The four symmetric pseudolines from a given Y-subgraph.

Figure 3: A sequence of triangles tilted away from the y-axis

Figure 4: Homotopies that reduce the area of the region XY

Figure 5: How a sequences of tilted triangles ends.

Figure 6: Analyzing which triangles contain portions of the T-curve.

Figure 7: Degree 7 schemes realized as T-curves.

Figure 8: T-curves of degree 8 with schemes $\langle\alpha\rangle\,\langle 1\langle\beta\rangle\rangle$.

Figure 9: T-curves of degree 8 with schemes $\langle\alpha\rangle\,\langle 1\langle 1\rangle\rangle\,\langle 1\langle\beta\rangle\rangle$.

Figure 10: T-curves of degree 8 with schemes $\langle\alpha\rangle\,\langle 1\langle\langle\beta\rangle\,\langle 1\langle 1\rangle\rangle\rangle\rangle$.

Figure 11: Triangulations of the Delaunay and placings family.

Figure 12: Proof of regularity for the 7H2 and 8B triangulation.

Figure 13: Proof of regularity for the H4 triangulation.