MAXIMAL CAPACITY, ROBIN CAPACITY, AND MINIMUM ENERGY

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Abstract. We present physical interpretations of maximal capacity and Robin capacity, and give an application of maximal capacity to the capacity of doubly-connected domains.

1. Introduction

Maximal capacity and Robin capacity arise as solutions to extremal problems in conformal mapping for logarithmic capacity. In this paper, we present minimum energy definitions of maximal capacity and Robin capacity (Theorems 1 and 4), and interpret these quantities physically in terms of single-layer and double-layer charge distributions. We also present an application of maximal capacity to geometric function theory by giving an analog of "conformal radius" for doubly-connected domains (Theorem 2).

2. Background and results

The logarithmic capacity of a compact set K in the plane is defined by considering the family of Borel measures

$$\mathcal{M} = \{\nu \geq 0 : support(\nu) \subset K, \quad \nu(K) = 1\}.$$

Physically, we think of K as generating a cylinder of conducting material extending infinitely in both directions from the z-plane. For any Borel set E, we think of $\nu(E)$ as the total charge per unit length along the cylinder generated by E, taken to be uniform along each orthogonal wire piercing the plane. Each $\nu \in \mathcal{M}$ induces a *logarithmic potential*

$$p_\nu(z) = \int_K \log \frac{1}{|z-\zeta|}\, d\nu(\zeta)$$

for points in the plane: the difference $p_\nu(z_1) - p_\nu(z_2)$ measures the amount of work it takes to move a positive charge between z_1 and z_2 against the electric field generated by ν. The *energy* of the distribution is

$$I(\nu) = \int p_\nu\, d\nu = \iint \log \frac{1}{|z-\zeta|}\, d\nu d\nu.$$

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The logarithmic capacity $d(K)$ is then defined to be e^{-V}, where V is the *minimum energy* of K,

$$V = \inf_{\nu\in\mathcal{M}} I(\nu).$$

A classical result (see Tsuji [13], Chapter III, Sections 3 and 8, or Ahlfors [1], Section 2-2, for example) relates Green's function to the *equilibrium potential*, which is the potential due to the distribution for which the above infimum is attained.

Theorem A. *Let K be a compact set whose unbounded complementary component Ω is bounded by a finite number of regular Jordan curves, and let $\Gamma = \partial\Omega$. Let $g(z)$ be Green's function with pole at infinity for Ω, and define*

$$\gamma = \lim_{z\to\infty} \{g(z) - \log|z|\}.$$

For Borel sets E, define

$$\nu_0(E) = -\frac{1}{2\pi}\int_{E\cap\Gamma} \frac{\partial g}{\partial n}\,|dz|,$$

where n denotes the outer normal to Γ. Then $V = I(\nu_0) = \gamma$, and the unique equilibrium potential is

$$p_0(z) = \begin{cases} \gamma - g(z) & z\in\Omega \\ \gamma & z\in\mathbf{C}\setminus\Omega \end{cases}$$

The constant γ is called "Robin's constant" for Ω. The equilibrium distribution is an example of a "single-layer distribution," in which all of the charge is concentrated on the boundary. Thus the capacity of K is equal to the capacity of its outer boundary Γ.

It is clear by the conformal invariance of Green's function that the capacity of Γ is invariant under conformal maps f of Ω normalized so that $f'(\infty) = 1$. On the other hand, if $A\cup B$ is a partition of the components of Γ, then the capacity of the subset A will not, in general, be preserved by these maps. We define the *maximal capacity* and *minimal capacity* of the set A relative to Ω to be

$$\sup_f d(f(A)),$$
$$\inf_f d(f(A)),$$

respectively, where f ranges over all normalized conformal maps of Ω.

Investigations of maximal capacity appear in [11, 12], whereas Duren and others [3, 4, 5] have studied minimal capacity. In particular, Duren and Schiffer [5] showed that minimal capacity is equal to Robin capacity, which is calculated from *Robin's function*, a generalization of Green's function. Robin's function $R(z)$ with pole at infinity is the function harmonic in Ω except at infinity, and having the properties

1. $R(z) - \log|z|$ is harmonic near infinity;
2. $R(z) = 0$ for $z\in A$; and
3. $\frac{\partial R}{\partial n}(z) = 0$ for $z\in B$.

Robin capacity is defined as $\delta(A) = e^{-\rho}$, where

$$\rho = \lim_{z\to\infty}\{R(z) - \log|z|\}.$$

In [11] the author obtained the following result for calculating maximal capacity, analogous to Duren and Schiffer's result.

Theorem B. *Let Ω be a finitely connected Jordan domain containing the point at infinity, with regular boundary Γ, and let $A \cup B$ be a partition of the components of Γ. Suppose B has N components, and let $h_i(z)$ denote the harmonic measure of the i-th component B_i relative to Ω. Let*

$$M(z) = g(z) + \sum_{i=1}^{N} c_i h_i(z),$$

where the c_i are uniquely determined by requiring

$$\frac{1}{2\pi} \int_{B_i} \frac{\partial M}{\partial n} \, |dz| = 0, \quad i = 1, \ldots, N. \tag{1}$$

Set $d_M(A) = e^{-\mu}$, where

$$\mu = \lim_{z \to \infty} \{M(z) - \log |z|\} = \gamma + \sum_{i=1}^{N} c_i h_i(\infty).$$

Then

$$d_M(A) = \sup_f d(f(A)),$$

where f ranges over all normalized conformal maps of Ω.

The goal of this paper is to show that the functions $R(z)$ and $M(z)$ can be viewed as equilibrium potentials, or minimum energy solutions to problems in electrostatics, in the same way Green's function is related to logarithmic capacity. We show that $M(z)$ arises as the solution to a problem for charged and uncharged conductors, and we reinterpret an existing minimum energy result for Robin capacity in terms of so-called "double-layer" charge distributions.

3. Maximal capacity and minimum energy

The minimum energy interpretation that we will derive for maximal capacity is the following.

Theorem 1. *Let Ω be a finitely connected Jordan domain containing the point at infinity, with regular boundary Γ, and let $A \cup B$ be a partition of the components of Γ. Let K, $K(A)$ and $K(B)$ be the compact sets bounded by Γ, A and B, respectively, with $K(B_i)$ bounded by the i-th component B_i of B. Consider the family of signed measures supported on K,*

$$\mathcal{M}_0 = \left\{ \nu = \nu^+ - \nu^- : \ \nu(K(A)) = 1, \quad \nu(K(B_i)) = 0, \quad i = 1, \ldots, m \right\}.$$

Let

$$V_M = \inf_{\nu \in \mathcal{M}_0} I(\nu).$$

Then there exists a unique distribution $\nu_M \in \mathcal{M}_0$ such that $V_M = I(\nu_M)$. The equilibrium distribution, defined for Borel sets E by

$$\nu_M(E) = -\frac{1}{2\pi} \int_{E \cap \Gamma} \frac{\partial M}{\partial n} \, |dz|,$$

satisfies $V_M = I(\nu_M) = \mu$, so that

$$d_M(A) = e^{-V_M}.$$

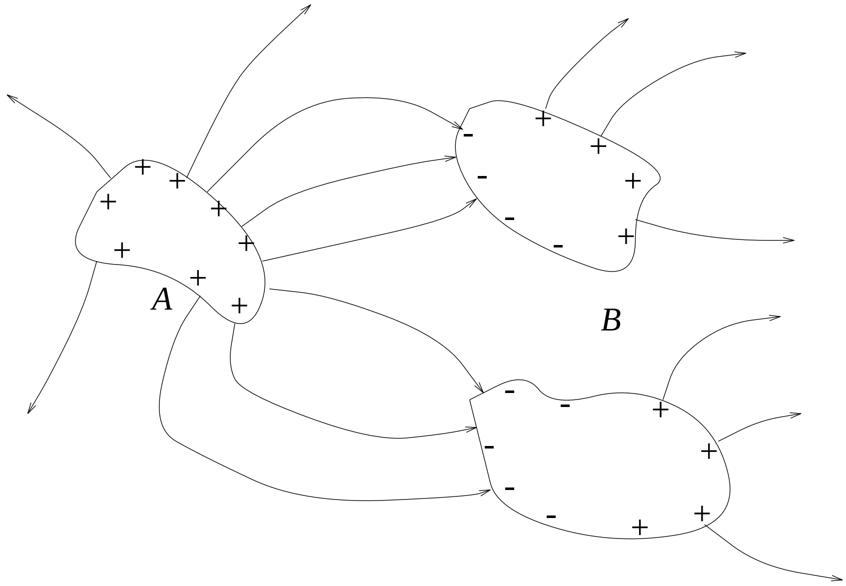

FIGURE 1. Flow-lines for the equilibrium distribution ν_M.

The equilibrium potential is defined at all points in the plane by

$$p_M(z) = \begin{cases} \mu - M(z), & z \in \Omega \\ \mu - c_i, & z \in K(B_i) \\ \mu, & z \in K(A) \end{cases} . \tag{2}$$

We think physically of placing a unit positive charge on the conductor $K(A)$, and leaving the conductors $K(B_i)$ neutrally charged. When the entire system comes to equilibrium (we imagine the components of $K(A)$ connected by thin wires to allow the charge to move freely between them), the charge on $K(A)$ will have polarized the charges on $K(B)$, further altering the electric field. The net charge on each component $K(B_i)$ is still zero, but the charge density could be positive or negative as we move around the component. The values of the equilibrium potential (2) reflect the fact that each conductor $K(B_i)$ lies on its own equipotential surface, as one would expect. This forces the flow-lines to enter and exit the components orthogonally. Moreover, the equilibrium distribution ν_M contains only positivie charges on A, a fact which follows from the result, proven in [11], that each c_i is positive, whence $\partial M/\partial n \leq 0$ on A. See Figure 1.

The proof of Theorem 1 closely follows Ahlfors' proof in [1] of Theorem A. A key lemma in that proof involves the *mixed energy*,

$$I(\nu_1, \nu_2) = \iint \log \frac{1}{|z - \zeta|} \, d\nu_1(\zeta) d\nu_2(z),$$

of two charge distributions.

Lemma (Ahlfors). *Let $\nu_1 = \nu_1^+ - \nu_1^-$ and $\nu_2 = \nu_2^+ - \nu_2^-$ be signed mass distributions on a compact set K with $\nu_1(K) = \nu_2(K)$ and $I(\nu_i^+) < \infty$, $I(\nu_i^-) < \infty$ for $i = 1, 2$. Then*

$$I(\nu_1) + I(\nu_2) - 2I(\nu_1, \nu_2) \geq 0.$$

The proof of the lemma, originally stated for positive charge distributions only, extends easily to signed distributions.

Proof of Theorem 1. Green's formula and (1) show that $\nu_M \in \mathcal{M}_0$. Moreover, Green's formula can be applied at every point in the plane to the resulting potential,

$$p_M(z) = -\frac{1}{2\pi}\int_\Gamma \log\frac{1}{|z-\zeta|}\frac{\partial M}{\partial n}\,|d\zeta|,$$

to show that (2) holds. And if $\nu \in \mathcal{M}_0$,

$$\begin{aligned} I(\nu_M,\nu) &= \int_{K(A)} p_M(z)d\nu(z) + \sum_i \int_{K(B_i)} p_M(z)\,d\nu(z) \\ &= \mu\int_{K(A)} d\nu + \sum_{i=1}^N (\mu - c_i)\int_{K(B_i)} d\nu \\ &= \mu. \end{aligned}$$

Thus

$$\mu = I(\nu_M) = I(\nu_M,\nu)$$

for any $\nu \in \mathcal{M}_0$. Ahlfors' lemma then implies $I(\nu) \geq I(\nu_M)$ for any $\nu \in \mathcal{M}_0$, so that ν_M is indeed an equilibrium distribution.

As to uniqueness, we may assume that the diameter of K is less than one, since there is a one-to-one correspondence between the admissible distributions for a set and the admissible distributions for its dilation. Suppose, then, that $\nu \in \mathcal{M}_0$ satisfies $I(\nu) = I(\nu_M)$. As above, $I(\nu_M,\nu) = I(\nu_M)$, so

$$I(\nu_m - \nu) = I(\nu_M) + I(\nu) - 2I(\nu_M,\nu) = 0.$$

Thus we need only show that for a signed measure $\sigma = \sigma^+ - \sigma^-$ supported on K,

$$I(\sigma) = 0 \quad\Longrightarrow\quad \sigma \equiv 0.$$

We have

$$\begin{aligned} 0 &= I(\sigma) = I(\sigma^+ - \sigma^-) = I(\sigma^+) + I(\sigma^-) - 2I(\sigma^+,\sigma^-) \\ &= I(\sigma^+) + I(\sigma^-), \end{aligned}$$

since the support of σ^+ is disjoint from the support of σ^-. But since $|z-\zeta| < 1$ for $z,\zeta \in K$,

$$I(\sigma^\pm) = \iint \log\frac{1}{|z-\zeta|}\,d\sigma^\pm d\sigma^\pm \geq 0,$$

with equality if and only if $\sigma^\pm$ are the zero measures. □

4. An application of maximal capacity to doubly-connected domains

Before moving on to the minimum energy interpretation of Robin capacity, we include the following application of maximal capacity to the logarithmic capacity of doubly-connected domains.

The results in [11] from which Theorem B is taken give the following geometric characterization for extremal domains maximizing the capacity of a subset A.

Theorem C. *Let Ω be a domain containing the point at infinity whose boundary components have been partitioned into subsets A and B. Let $F(z)$ be a normalized conformal map of Ω. Then*

$$d(F(A)) = d_M(A)$$

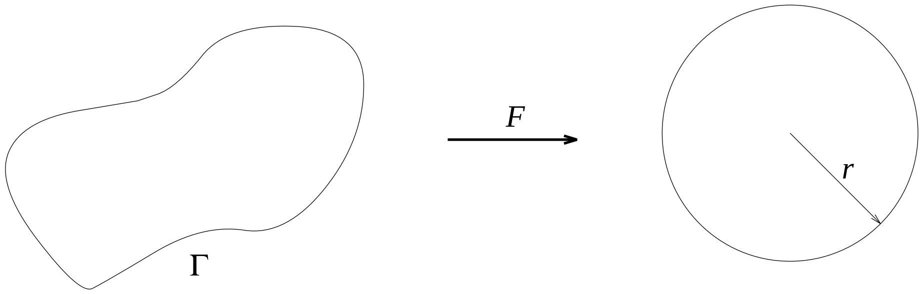

FIGURE 2. $d(\Gamma) = r$.

if and only if each component $F(B_i)$ *of* $F(B)$ *is a union of analytic arcs on which Green's function* $\widetilde{g}(z)$ *for the larger domain* $\widetilde{F(\Omega)}$ *bounded only by* $F(A)$ *is constant. In this case,* $\widetilde{g}(z) = c_i$ *on* B_i*, where* c_i *is uniquely defined by* (1).

Thus in the special case A is a single component of the boundary of Ω, one extremal function maximizing the capacity of A maps A to a circle, and each of the other components to a circular slit, concentric with and exterior to the circle.

We use these results to give an analog of "conformal radius" for doubly connected domains. The "conformal radius" of a simply connected domain containing infinity is defined to be the radius of the disk whose exterior can be mapped onto the domain by a normalized conformal map (see Figure 2). It is clear that this is the same as the capacity of the boundary, since Green's function for the exterior of a disk of radius r is $\log \frac{|z|}{r}$.

In theory the capacity of a doubly connected domain can be computed when a conformal map of the domain onto an annulus is given. One can express Green's function for an annulus explicitly in terms of elliptic theta functions (see [6], pp. 259–263), and thus obtain Robin's constant for the domain. But there are no elementary expressions for the capacity of, say, the union of a disk with a circular arc in terms of the geometry of the components. Our result gives the capacity of the boundary of a doubly connected domain in terms of the radii of the circles and circular slits of the two conformally equivalent circular slit domains, where each component gets its chance to be mapped to a full circle (see Figure 3). We use this to obtain an estimate for the capacity of the union of a disk with a circular arc.

Theorem 2. *Let* Ω *be a domain containing the point at infinity whose boundary* Γ *consists of two components* A *and* B*. Let* $F_A(z)$ *(respectively,* $F_B(z)$*) be a normalized conformal map of* Ω *such that* $F_A(A)$ *(respectively,* $F_B(B)$*) is a circle, and* $F_A(B)$ *(respectively,* $F_B(A)$*) is an exterior concentric circular slit. Let* r_A *(respectively,* r_B*) be the radius of the circle* $F_A(A)$ *(respectively,* $F_B(B)$*), and* R_A *(respectively,* R_B*) be the radius of the slit* $F_A(B)$ *(respectively,* $F_B(A)$*). Then*

$$d(\Gamma) = r_A \exp\left\{ \frac{\left(\log \frac{R_A}{r_A}\right)^2}{\log \frac{R_A R_B}{r_A r_B}} \right\}. \tag{3}$$

Proof. Since F_A is extremal for maximizing the capacity of A, Theorem B implies

$$r_A = d_M(A) = d(\Gamma) e^{-c_A h_B(\infty)}, \tag{4}$$

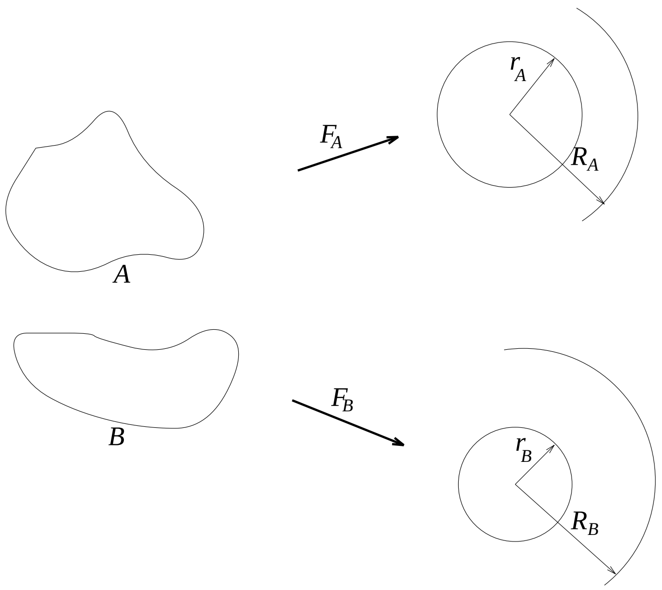

FIGURE 3. $d(A \cup B) = r_A \exp\left\{\left(\log \frac{R_A}{r_A}\right)^2 / \log \frac{R_A R_B}{r_A r_B}\right\}$.

where $h_B(z)$ is the harmonic measure of B relative to Ω. It's clear that $\widetilde{g}_A(z) = \log \frac{|z|}{r_A}$ is Green's function for the domain bounded only by $F_A(A)$, and the value of $\widetilde{g}_A(z)$ on the slit $F_A(B)$ is

$$c_A = \log \frac{R_A}{r_A}.$$

Define c_B similarly in terms of the map F_B – it has the value $\log \frac{R_B}{r_B}$.

In [11] we use (1) to show that the constants c_i in Theorem B can be expressed in terms of the values of the harmonic measures at infinity, and the periods of the harmonic conjugates of the harmonic measures. As applied to our case the results imply

$$\begin{aligned} c_A &= \frac{h_B(\infty)}{p_{BB}} \\ c_B &= \frac{h_A(\infty)}{p_{AA}}, \end{aligned}$$

where p_{AA} (respectively, p_{BB}) is the the period of the harmonic conjugate of h_A about A (respectively, h_B about B). But $h_A(\infty) + h_B(\infty) = 1$, and $p_{AA} = p_{BB}$. Thus $c_A + c_B = 1/p_{BB}$, and

$$\begin{aligned} c_A h_B(\infty) &= c_A^2 p_{BB} = \frac{c_A^2}{c_A + c_B} \\ &= \frac{\left(\log \frac{R_A}{r_A}\right)^2}{\log \frac{R_A R_B}{r_A r_B}}, \end{aligned}$$

which together with (4) proves the theorem. $\square$

But equation (3) must hold with the roles of A and B reversed. We set the expressions with A and B reversed equal to one another, then take the logarithm of both sides and expand all of the logarithmic terms to find

$$(\log R_A - \log R_B)(\log R_A + \log R_B) = (\log R_A - \log R_B)(\log r_A + \log r_B).$$

Thus either $R_A = R_B$, or $R_A R_B = r_A r_B$. But by definition, $R_A > r_A$, and $R_B > r_B$, so we must have $R_A = R_B$. We record this fact concerning the geometry of circular slit mappings for doubly connected domains as a corollary.

Corollary 1. *Let Ω be a doubly connected domain containing the point at infinity, with boundary components A and B. Let $F_A(z)$ be a normalized conformal map of Ω taking A to a circle and B to an exterior concentric circular slit. Let $F_B(z)$ be a normalized conformal map of Ω taking B to a circle and A to an exterior concentric circular slit. Then the radius of the circular slit $F_A(B)$ equals the radius of the circular slit $F_B(A)$.*

Finally, recall that the capacity of a circular arc of angle ψ and radius R is $R\sin(\psi/4)$. We see by applying this to the arc $F_A(B)$ that

$$r_B = d_M(B) = d(F_B(B)) > d(F_A(B)) = R_A \sin\frac{\psi}{4}.$$

The inequality is strict by the characterization of extremal domains given in Theorem C. On the other hand we clearly have $r_B < R_B = R_A$. We apply these results to obtain the following estimate for the capacity of a disk union a circular arc.

Corollary 2. *Let K be the union of a disk of radius r and a concenctric circular arc of angle ψ and radius R. Then*

$$r \exp\left\{ \frac{\left(\log \frac{R}{r}\right)^2}{\log \frac{R}{r\sin\frac{\psi}{4}}} \right\} < d(K) < R.$$

Note: Another generalization of conformal radius for multiply connected domains was introduced by Hersch [7]. His "harmonic radius" is defined by letting the singularity for Green's function range over all points of the domain. See [2] for a survey of this topic.

5. Robin capacity and minimum energy

In recent work [4], we have shown that Robin capacity has a minimum energy property relative to a "Neumann kernel" rather than a logarithmic kernel. Let $\widehat{\Omega}$ denote the larger domain bounded only by B, and let $\zeta \in \widehat{\Omega}$. The "Neumann function" $N(z,\zeta)$ of $\widehat{\Omega}$ is the function harmonic in $\widehat{\Omega}$ except at ζ and infinity, with the properties

1. $N(z,\zeta) - \log\frac{1}{|z-\zeta|}$ is harmonic near ζ;
2. $N(z,\zeta) + \log|z|$ is harmonic near infinity;
3. $\lim_{z\to\infty}\{N(z,\zeta) + \log|z|\} = 0$; and
4. $\frac{\partial N}{\partial n}(z,\zeta) = 0$ for $z \in B$.

In addition, $N(z,\zeta)$ has the symmetry property $N(z,\zeta) = N(\zeta,z)$ for z and ζ in $\widehat{\Omega}$.

The main result from [4] is as follows.

Theorem D. *Let $N(z,\zeta)$ be the Neumann function of the domain $\widehat{\Omega}$ bounded only by B. Suppose that $\nu \geq 0$ is a unit charge distribution on the compact set $K(A)$ with boundary A, and let*

$$U_\nu(z) = \int_{K(A)} N(z,\zeta)\, d\nu(\zeta) \tag{5}$$

be its potential with respect to the Neumann kernel. Let $W = \inf_\nu J(\nu)$, where

$$J(\nu) = \iint N(z,\zeta) d\nu(z)\, d\nu(\zeta)$$

is the energy with respect to $N(z,\zeta)$. Then $W = \rho$, where the Robin capacity of A with respect to Ω is $\delta(A) = e^{-\rho}$, and there exists a unique unit measure ν such that $J(\nu) = \rho$. The equilibrium potential has the property that $U_\nu(z) = \rho$ quasi-everywhere on $K(A)$ and equals, in fact, $\rho - R(z)$ where $R(z)$ is Robin's function for A relative to Ω.

Our aim is to interpret the "Neumann potentials" (5) physically in terms of double-layer charge distributions, which we now review briefly.

5.1. **Double-layer potentials and energy.** A double-layer potential on a regular Jordan curve S arises as the limit of the potential due to equal but opposite charge distributions separated by a small distance. Let S_ϵ be a copy of S lying at a distance ϵ to one side of S. Let a positive charge distribution on S be given by a density function $D_\epsilon(z)$. Define $D'_\epsilon(z')$ on S' so that $D'_\epsilon(z') = -D_\epsilon(z)$, where $z \in S$ lies in the normal direction from $z' \in S'$. If we let $D_\epsilon(z)$ increase to infinity in such a way that $\lim_{\epsilon\to 0} \epsilon D_\epsilon(z) = D(z)$, uniformly in z for some function $D(z)$, the limiting potential is calculated to be

$$V(z) = \int_S D(\zeta) \frac{\partial}{\partial n} \log \frac{1}{|z-\zeta|}\, |d\zeta|. \tag{6}$$

$V(z)$ is called a "double-layer potential" due to the "double-layer density" function $D(z)$. The basic results concerning double-layer potentials are summarized in the following theorem (see Sternberg and Smith [10], Chap. V, Art. 5 - 8, or Kellogg [8], Chap. 6, Sections 6 and 7, for example).

Theorem E. *Let the real function $D(z)$ be continuous on the regular Jordan curve S. Then the double-layer potential (6) exists at all points $z \in \mathbf{C}$ and is harmonic in $\mathbf{C} \setminus S$. At each point $z \in S$ with outer normal direction n, the one-sided limits $V^+(z)$ and $V^-(z)$ along n and $-n$, respectively, exist and satisfy*

$$\begin{aligned} V^+(z) - V^-(z) &= 2\pi D(z), \\ V^+(z) + V^-(z) &= 2V(z). \end{aligned}$$

If $D(z)$ has a continuous second derivative along S, the one-sided limits

$$\left(\frac{\partial V}{\partial n}\right)^+ (z), \quad \text{and} \quad \left(\frac{\partial V}{\partial n}\right)^- (z) \tag{7}$$

exist and are equal.

As Henrici shows in [6], Section 15.8, the discontinuity properties of V can be derived from similar properties of Cauchy integrals, since V can be expressed as the real part of a Cauchy integral,

$$V(z) = \mathbf{Re}\left\{\frac{1}{2\pi i}\int_S \frac{2\pi D(\zeta)}{\zeta - z}\,d\zeta\right\}. \tag{8}$$

Double-layer potentials occur naturally in the representation of harmonic functions. If $u(z)$ is harmonic in the unbounded complement of S, with a logarithmic singularity at infinity, and normalized so that $u(z) - \log|z| \to 0$ as $z \to \infty$, Green's formula shows that

$$u(z) = \frac{1}{2\pi}\int_S \log\frac{1}{|z-\zeta|}\frac{\partial u(\zeta)}{\partial n}\,|d\zeta| - \frac{1}{2\pi}\int_S u(\zeta)\frac{\partial}{\partial n}\log\frac{1}{|z-\zeta|}\,|d\zeta| \tag{9}$$

Thus any harmonic function of the above type can be interpreted as the sum of potentials of a single and a double-layer.

We can calculate the energy I due to the double-layer density $D(z)$ on S by using the same limiting process used to calculate the potential, with the result that

$$I(D) = \int_S\int_S D(\zeta)D(z)\frac{\partial^2}{\partial n(z)\partial n(\zeta)}\log\frac{1}{|z-\zeta|}\,|d\zeta||dz| = \int_S D(z)\frac{\partial V}{\partial n}\,|dz|.$$

The normal derivative above is to be understood as the limit (7).

5.2. **Robin capacity and the double-layer potential.** Our goal is to interpret Robin's function as the equilibrium potential due to a combination of single and double-layer distributions. A positive charge distribution ν on $K(A)$, together with a double-layer density $D(z)$ on B. induces the *mixed potential*

$$V_{\nu,D}(z) = \int_B D(\zeta)\frac{\partial}{\partial n}\log\frac{1}{|z-\zeta|}\,|d\zeta| + \int_{K(A)}\log\frac{1}{|z-\zeta|}\,d\nu(\zeta).$$

The energy is calculated to be

$$\begin{aligned} I(\nu, D) &= \int_B\int_B D(\zeta)D(z)\frac{\partial^2}{\partial n\partial n}\log\frac{1}{|z-\zeta|}\,|d\zeta||dz| \\ &+ 2\int_{K(A)}\int_B D(z)\frac{\partial}{\partial n(z)}\log\frac{1}{|z-\zeta|}\,|dz|d\nu(\zeta) \\ &+ \int_{K(A)}\int_{K(A)}\log\frac{1}{|z-\zeta|}\,d\nu d\nu, \end{aligned}$$

which reduces easily to

$$I(\nu, D) = \int_B \frac{\partial V_{\nu,D}}{\partial n}D(z)\,|dz| + \int_{K(A)} V_{\nu,D}\,d\nu. \tag{10}$$

We first show that mixed potentials can be constructed to mimic the key property of Robin's function: the vanishing of its normal derivative on the set B.

Theorem 3. *For every positive unit charge distribution ν on $K(A)$, there exists a double-layer density $D_\nu(z)$ on B such that the mixed potential $V_{\nu,D_\nu}(z)$ satisifes*

$$\frac{\partial V_{\nu,D_\nu}}{\partial n} = 0 \tag{11}$$

on B. The potential $V_{\nu,D_\nu}(z)$ equals the Neumann potential $U_\nu(z)$.

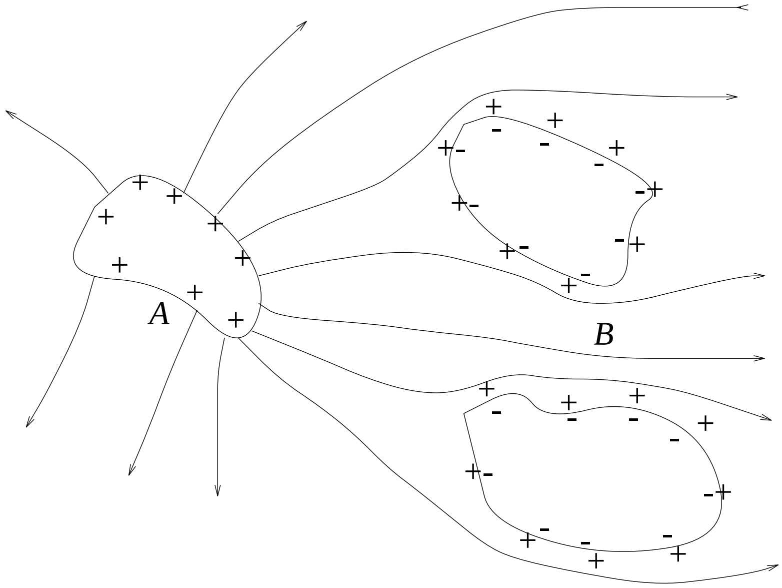

FIGURE 4. Flow-lines due to a charge distribution ν on A and the associated double-layer density D_ν on B.

Physically we can think of placing a unit positive charge on A, and then setting up a double-layer distribution on B so that the flow-lines of the resulting potential are parallel to B (see Figure 4).

Before proving the theorem we remark that the density $D_\nu(z)$ is not unique. Indeed, let $b(z)$ be any function analytic in the collection of bounded components whose boundary is B, continuous up to B, and real-valued on B (for example, define $b(z) \equiv b_i$ on B_i, for any $b_i \in \mathbf{R}$). Then the Cauchy integral

$$\frac{1}{2\pi i}\int_B \frac{b(\zeta)}{\zeta - z}\,d\zeta$$

vanishes for $z \in \widehat{\Omega}$. So if the mixed potential due to ν and $D_\nu(z)$ satisfies (11), the mixed potential due to ν and $D_\nu(z) + b(z)$ does also, by the Cauchy integral representation (8) for double-layer potentials.

Proof of Theorem 3. Let $\widehat{\Omega} \supset \Omega$ be the domain bounded only by B. Let $h(z, \widetilde{z})$ be the kernel for solving the "boundary value problem of the second kind" on B. (See Nehari [9], Chapter 1, Section 7, for example. Unfortunately, as in this reference, $h(z, \widetilde{z})$ is also commonly referred to as the "Neumann function" for $\widehat{\Omega}$.) This is the function harmonic in $\widehat{\Omega} \setminus \widetilde{z}$, and defined by the properties

1. $h(z, \widetilde{z}) - \log \frac{1}{|z - \widetilde{z}|}$ is harmonic near $\widetilde{z}$;
2. $\frac{\partial h}{\partial n(z)} =$ constant on B; and
3. $\int_B h(z, \widetilde{z})\,ds = 0$.

The constant in the second condition is uniquely determined by the other conditions. The final criterion is a normalization condition. In addition, $h(z, \widetilde{z})$ has the symmetry property $h(z, \widetilde{z}) = h(\widetilde{z}, z)$.

We use the reproducing properties of h to represent the function $N(z,\zeta) - \log\frac{1}{|z-\zeta|}$, which, for fixed $\zeta \in \widehat{\Omega}$, is harmonic in all of $\widehat{\Omega}$, and has normal derivative equal to $-\partial/\partial n(z)\log 1/|z-\zeta|$ on B. We find that

$$\begin{aligned} N(z,\zeta) &= \log\frac{1}{|z-\zeta|} - \frac{1}{2\pi}\int_B h(\widetilde{z},z)\frac{\partial}{\partial n(\widetilde{z})}\log\frac{1}{|\widetilde{z}-\zeta|}\,|d\widetilde{z}| \\ &= \log\frac{1}{|z-\zeta|} - \frac{1}{2\pi}\int_B h(\widetilde{z},\zeta)\frac{\partial}{\partial n(\widetilde{z})}\log\frac{1}{|\widetilde{z}-z|}\,|d\widetilde{z}|. \end{aligned}$$

For the last equality we have used the symmetry property of the function $N(z,\zeta) - \log\frac{1}{|z-\zeta|}$. Thus $N(z,\zeta)$ can be viewed as the potential at z due to a point charge at ζ and a double-layer density of $-\frac{1}{2\pi}h(\widetilde{z},\zeta)$ for $\widetilde{z}$ on B. We integrate both sides of the above equation with respect to $d\nu$ on $K(A)$ and see

$$\int_{K(A)} N(z,\zeta)\,d\nu(\zeta) = \int_{K(A)} \log\frac{1}{|z-\zeta|}\,d\nu(\zeta) + \int_B D_\nu(\widetilde{z})\frac{\partial}{\partial n(\widetilde{z})}\log\frac{1}{|\widetilde{z}-z|}\,|d\widetilde{z}|,$$

where

$$D_\nu(\widetilde{z}) = -\frac{1}{2\pi}\int_{K(A)} h(\widetilde{z},\zeta)\,d\nu(\zeta). \tag{12}$$

Thus

$$U_\nu(z) = \int_{K(A)} N(z,\zeta)\,d\nu(\zeta)$$

is the mixed potential $V_{\nu,D_\nu}(z)$, which moreover has normal derivative equal to zero on B, since $\partial N(z,\zeta)/\partial n(z) = 0$ on B. □

The energy (10) due to ν and D_ν reduces to

$$I(\nu,D_\nu) = \int_{K(A)}\int_{K(A)} N(z,\zeta)d\nu(z)\,d\nu(\zeta),$$

since the normal derivative of the potential vanishes on B (notice that the energy is not dependent on the choice of D_ν). Thus $I(\nu,D_\nu)$ is just $J(\nu)$, and we have established the equivalence of Theorem D with the following.

Theorem 4. *For any positive unit charge distribution ν on A, let D_ν be the double-layer density function* (12) *on B such that the mixed potential $V_{\nu,D_\nu}(z)$ has normal derivative equal to zero on B. Then*

$$\inf_{\nu\in\mathcal{M}} I(\nu,D_\nu) = \rho,$$

where $\delta(A) = e^{-\rho}$ is the Robin capacity of A with respect to Ω. The equilibrium potential is $\rho - R(z)$, where $R(z)$ is Robin's function for A relative to Ω.

By (9), we may write

$$\rho - R(z) = -\frac{1}{2\pi}\int_A \log\frac{1}{|z-\zeta|}\frac{\partial R(\zeta)}{\partial n}\,|d\zeta| + \frac{1}{2\pi}\int_B R(\zeta)\frac{\partial}{\partial n}\log\frac{1}{|z-\zeta|}\,|d\zeta|.$$

Thus the unique equilibrium charge distribution on A is given by

$$d\nu(\zeta) = -\frac{1}{2\pi}\frac{\partial R}{\partial n}(\zeta)|d\zeta|,$$

while a corresponding equilibrium double-layer density on B is

$$D_\nu(\zeta) = \frac{1}{2\pi}R(\zeta).$$

References

[1] L. Ahlfors, *Conformal invariants: topics in geometric function theory*, McGraw-Hill, New York, 1973.

[2] C. Bandle and M. Flucher, *Harmonic radius and concentration of energy; hyperbolic radius and Liouville's equations $\delta u = e^u$ and $\delta u = u^{(n+2)/(n-2)}$*, SIAM Review **38** (1996), 191–238.

[3] P. Duren and J. Pfaltzgraff, *Robin capacity and extremal length*, J. Math. Anal. Appl. **179** (1993), 110–119.

[4] P. Duren, J. Pfaltzgraff, and R. E. Thurman, *Physical interpretation and further properties of Robin capacity*, Algebra i Analiz (=St. Petersburg Math. J.), to appear.

[5] P. Duren and M. M. Schiffer, *Robin functions and distortion of capacity under conformal mapping*, Complex Variables Theory Appl. **21** (1993), 189–196.

[6] P. Henrici, *Applied and computational complex analysis*, vol. 3, Wiley-Interscience, 1986.

[7] J. Hersch, *Transplantation harmonique, transplantation par modules, et théorèmes isopérimétriques*, Comment. Math. Helv. **44** (1969), 354–366.

[8] O. Kellogg, *Foundations of potential theory*, Springer-Verlag, Berlin, 1929.

[9] Z. Nehari, *Conformal mapping*, McGraw–Hill, New York, 1952, Dover reprint (1975).

[10] W. J. Sternberg and T. L. Smith, *The theory of potential and spherical harmonics*, University of Toronto Press, Toronto, 1946.

[11] R. E. Thurman, *Upper bound for distortion of capacity under conformal mapping*, Trans. Amer. Math. Soc. **346** (1994), no. 2, 605–616.

[12] R. E. Thurman, *Bridged extremal distance and maximal capacity*, Pacific J. Math., to appear.

[13] M. Tsuji, *Potential theory in modern function theory*, Maruzen, Tokyo, 1959.

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